technische universität
dortmund

Translating Between
Wavelet Tree and Wavelet Matrix Construction

Patrick Dinklage

Efficient Rank and Select Queries
on binary alphabets

$$B = \texttt{10100011000}$$

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- constant time with $O(n)$ preprocessing time and $o(n)$ bits of space

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(*$rank_0$* and *$select_0$* analogously)

Efficient Rank and Select Queries
on integer alphabets

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$$T = \texttt{wavelettree}$$

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- $\Sigma = \{\texttt{a},\texttt{e},\texttt{l},\texttt{r},\texttt{t},\texttt{v},\texttt{w}\}$

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- time $O(lg\ \sigma)$ with $O(n\ lg\ \sigma)$ preprocessing time
  using a **wavelet tree** or **wavelet matrix**

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$\sigma := |\Sigma|$

- time $O(lg\ \sigma)$ with $O(n\ lg\ \sigma)$ preprocessing time
  using a **wavelet tree** or **wavelet matrix**

The Wavelet Tree

$\Sigma =$ {a,e,l,r,t,v,w}

$T =$ `wavelettree`

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The Wavelet Tree

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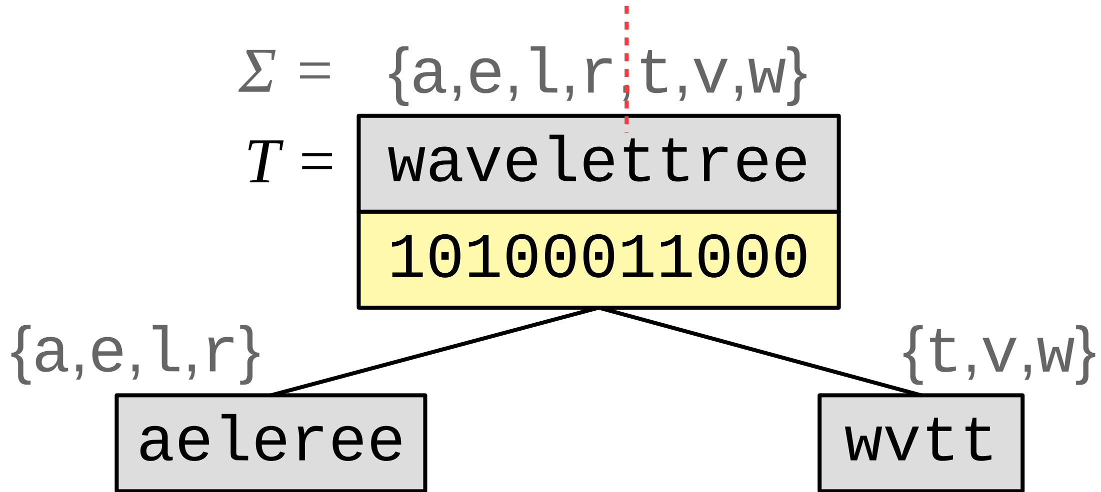
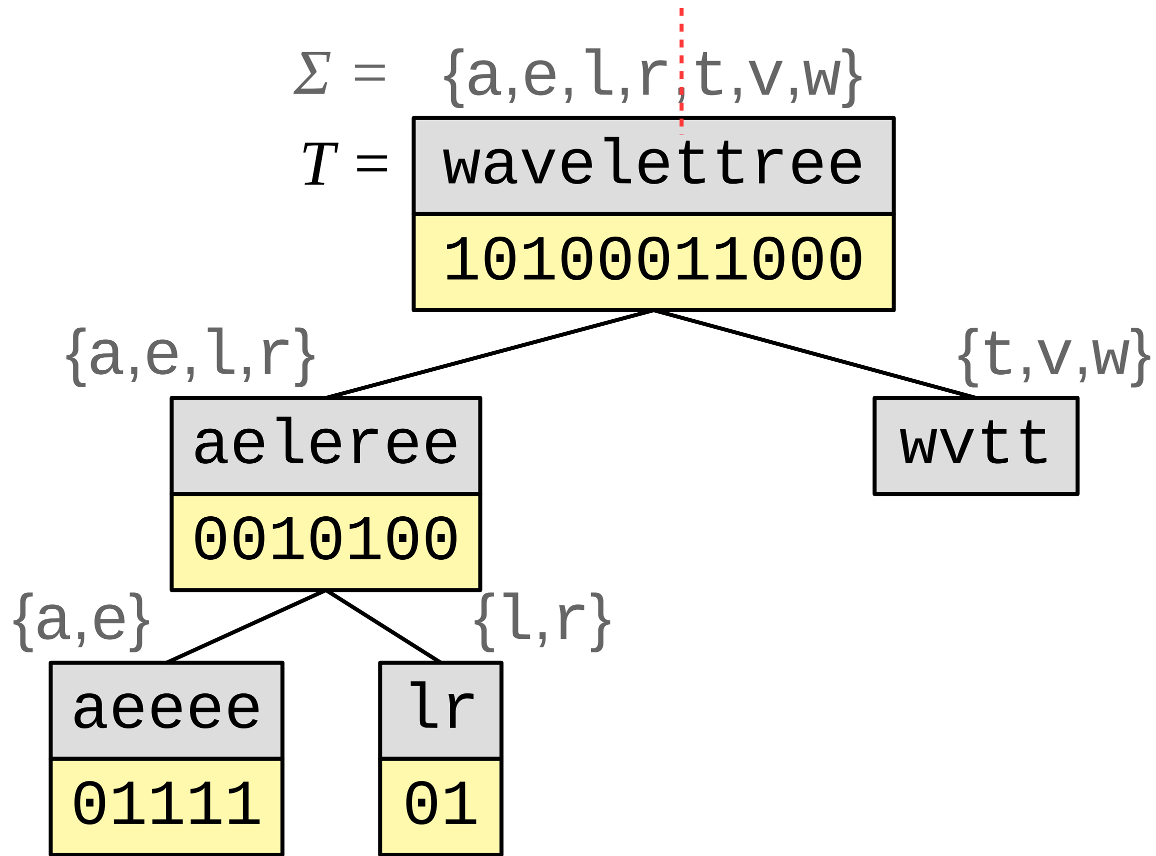
$T =$

wavelettree
10100011000

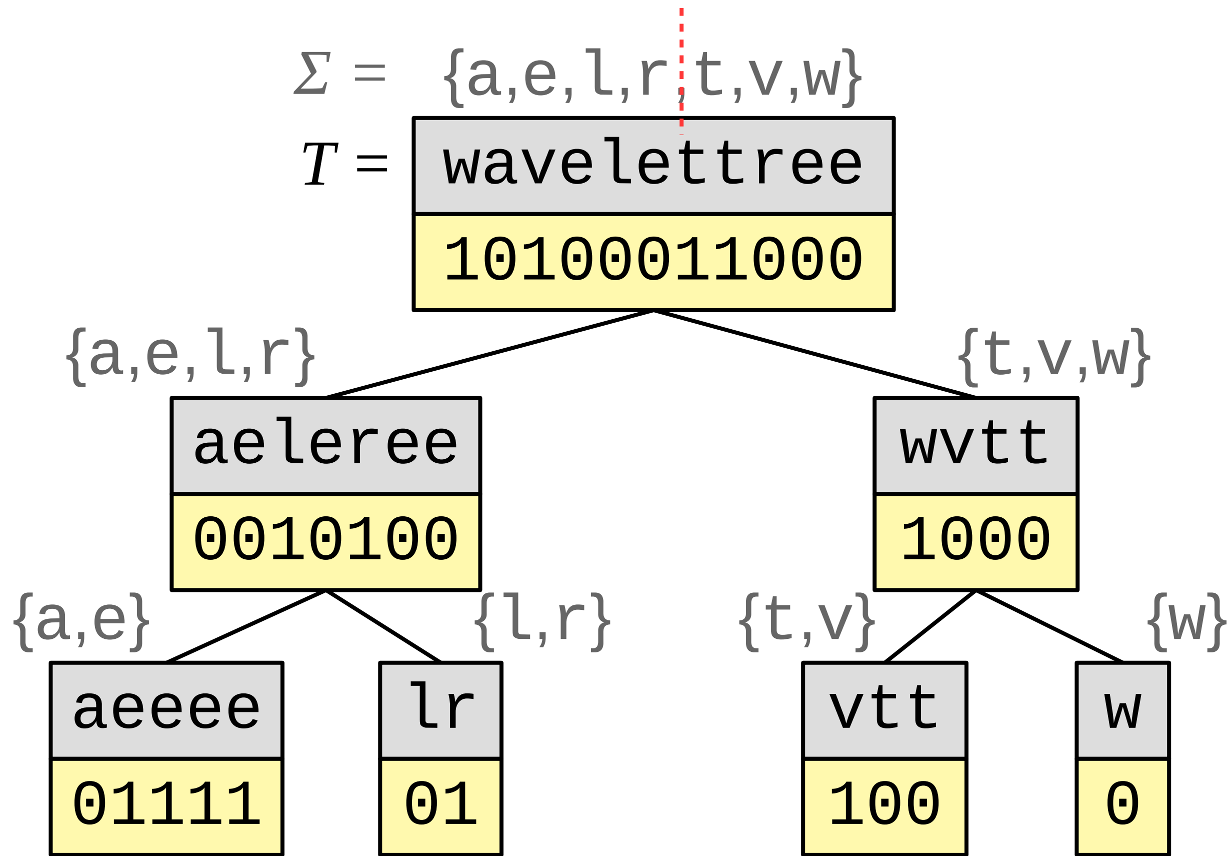
The Wavelet Tree



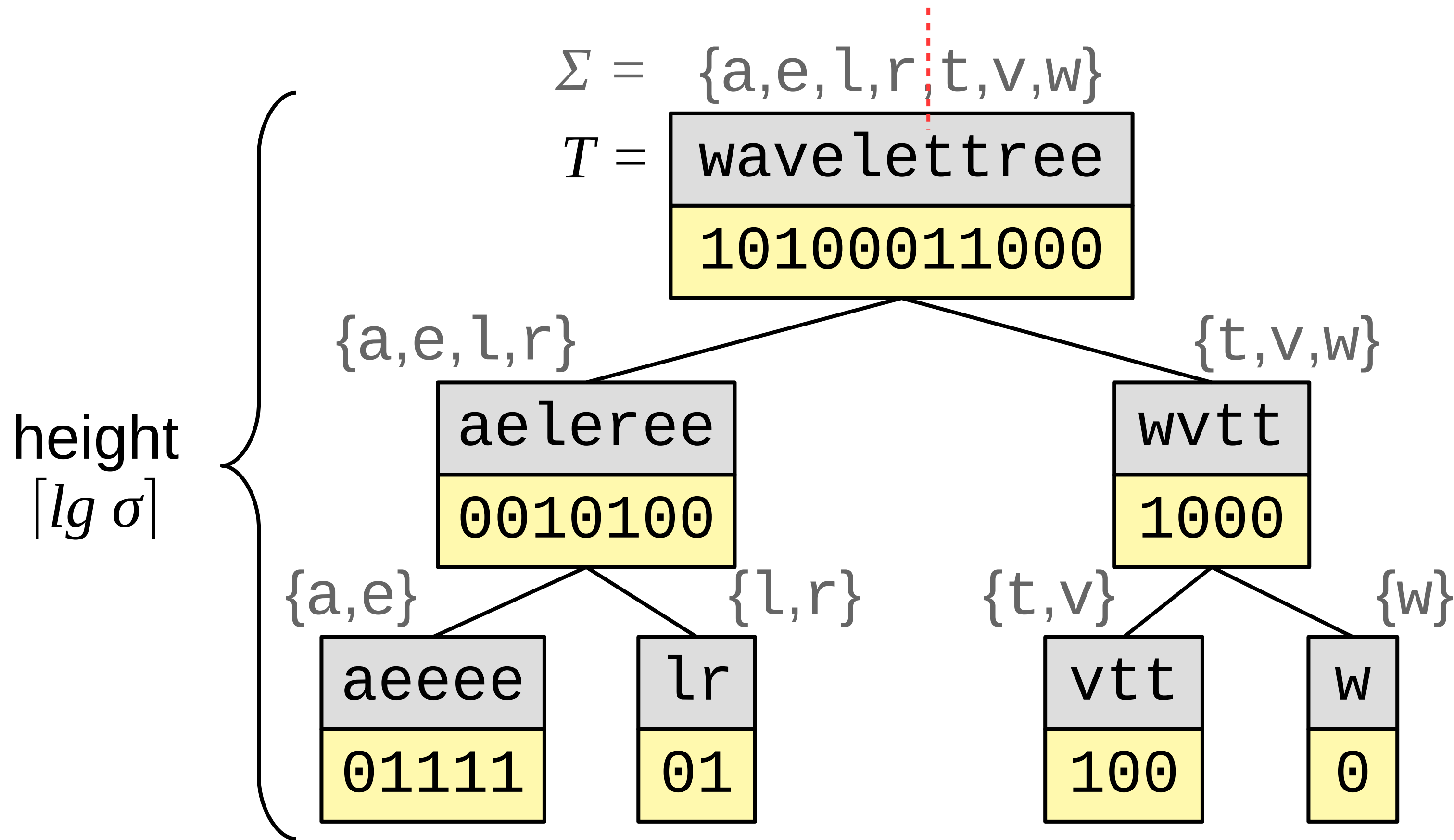
The Wavelet Tree



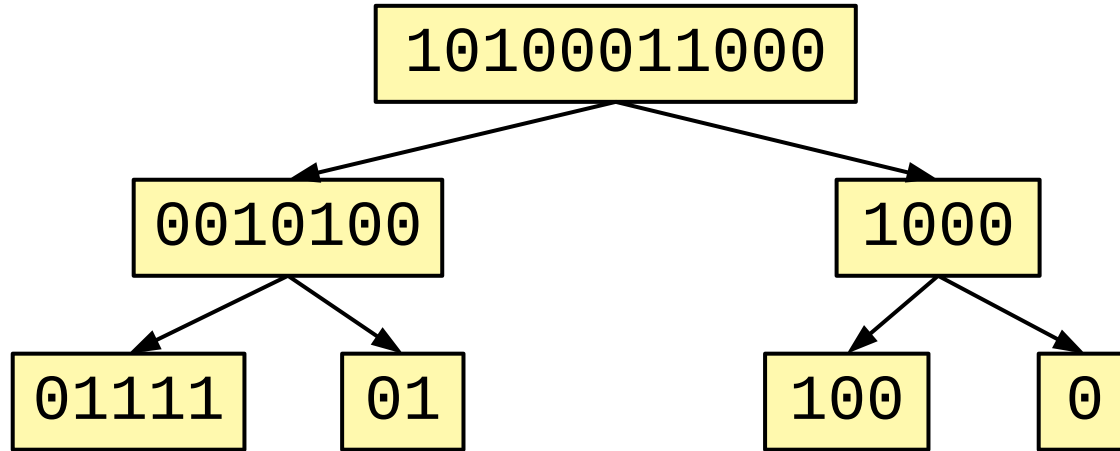
The Wavelet Tree



The Wavelet Tree

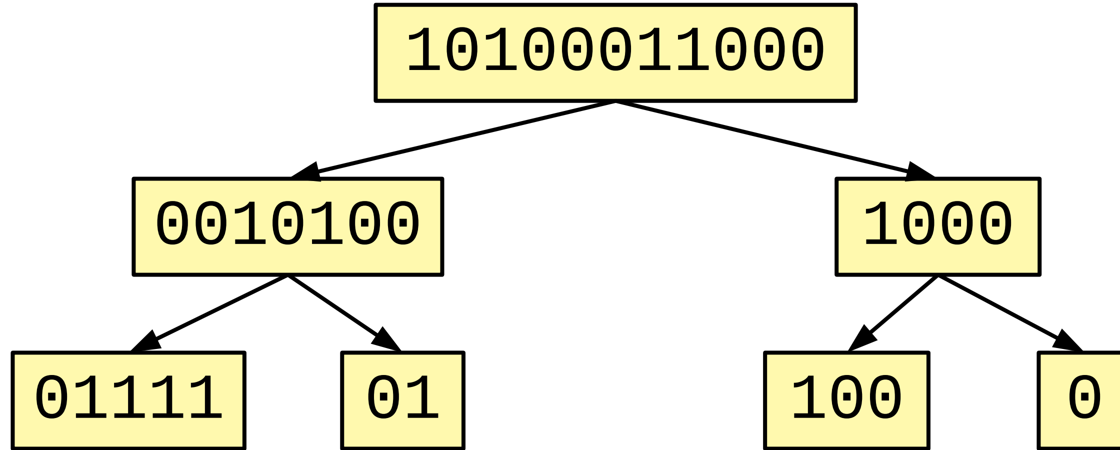


The Wavelet Tree
Storage



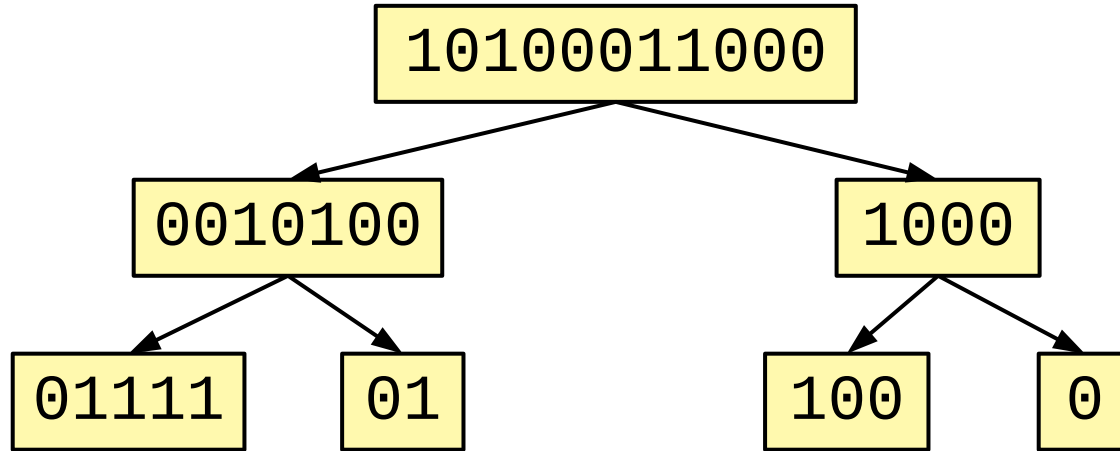
The Wavelet Tree

Storage



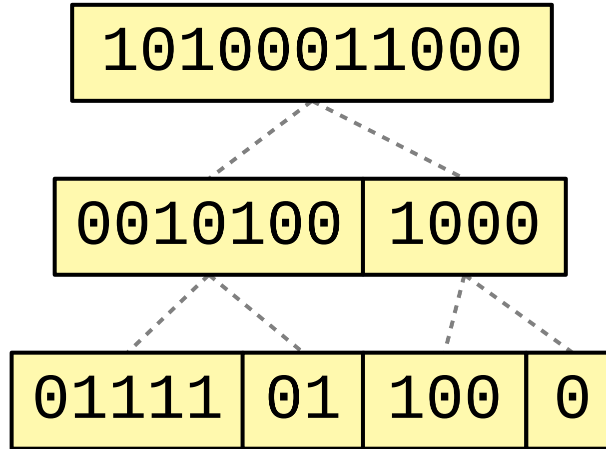
- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select

The Wavelet Tree
Storage



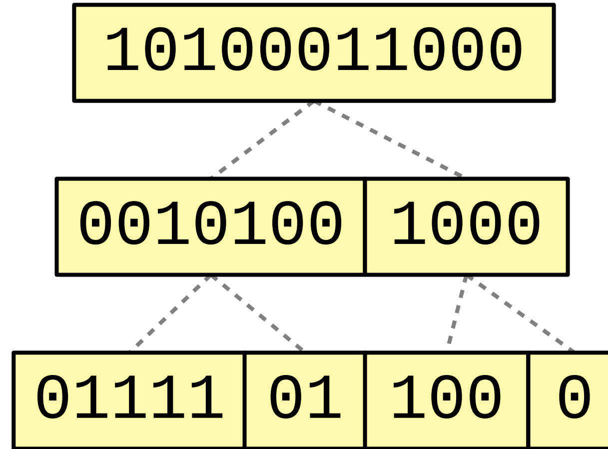
- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select
  - up to $(2\sigma - 2) \lceil lg\ n \rceil$ bits for pointers

The Wavelet Tree
Storage



- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select
- up to $(2\sigma - 2) \lceil lg\ n \rceil$ bits for node boundaries

The Wavelet Tree
Storage



- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select
- ~~up to $(2\sigma - 2) \lceil lg\ n \rceil$ bits for node boundaries~~

The Wavelet Tree
Storage

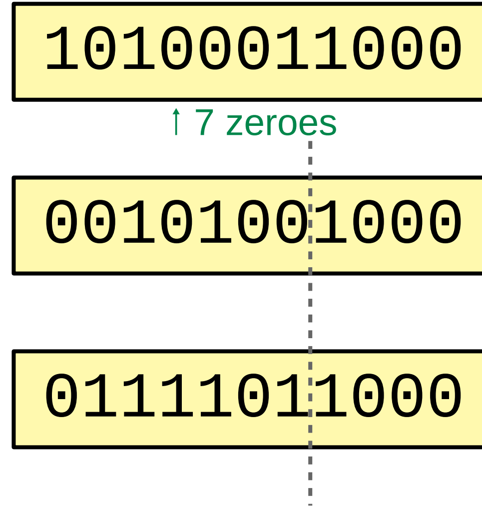
10100011000

00101001000

01111011000

- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select

The Wavelet Tree
Storage



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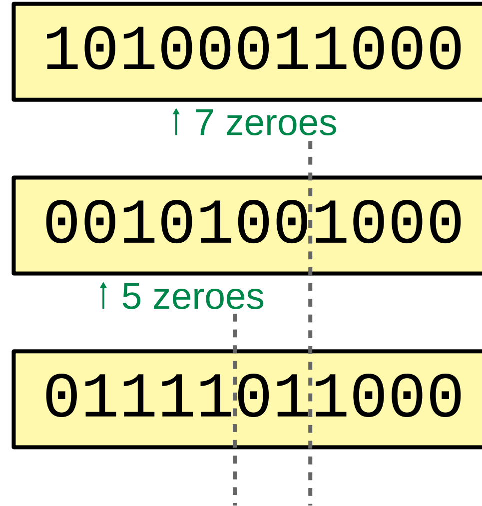

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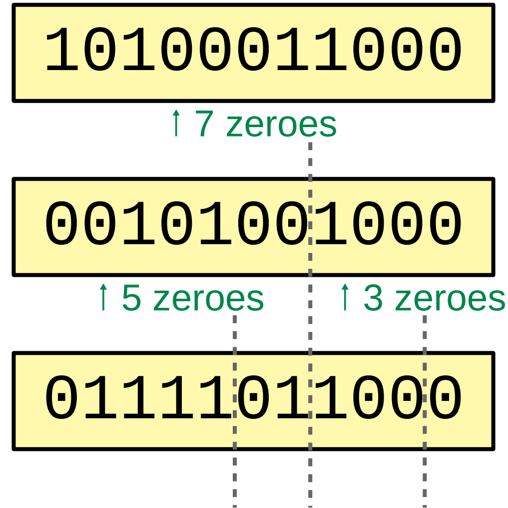

- $(n+o(n))\lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select

The Wavelet Tree
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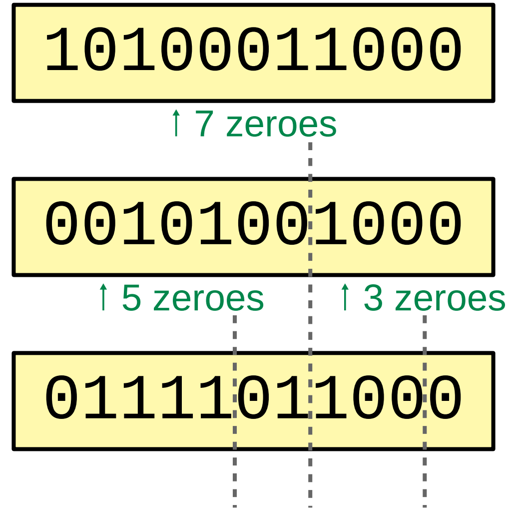
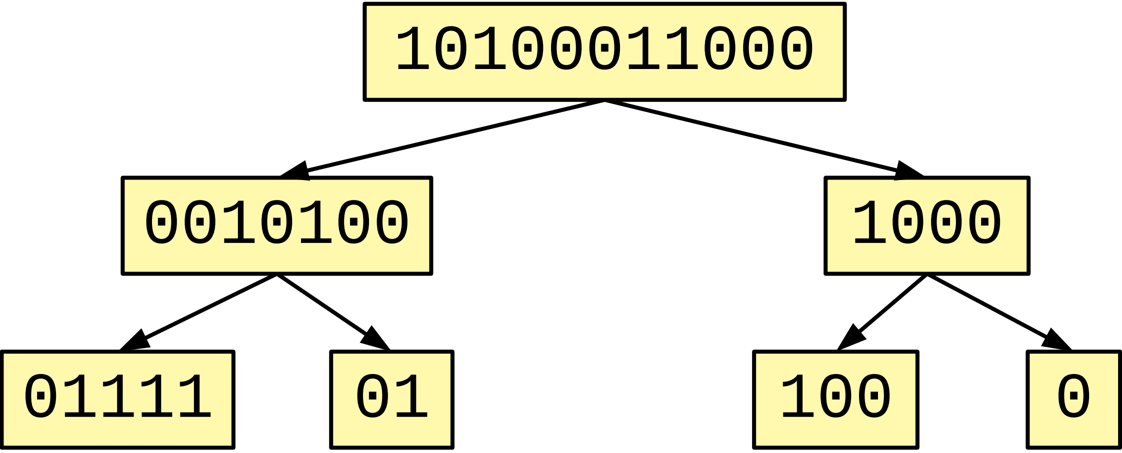
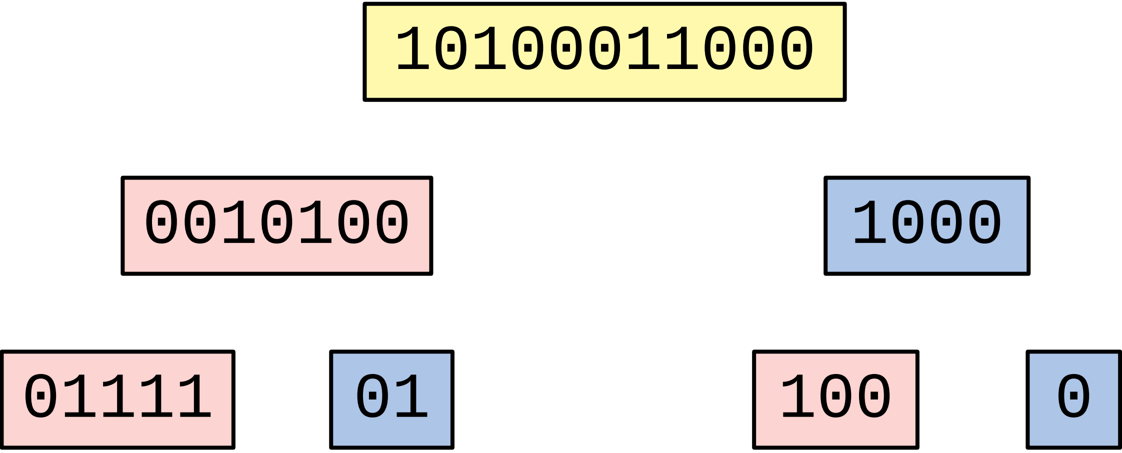


- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select
  - implicit node boundaries!

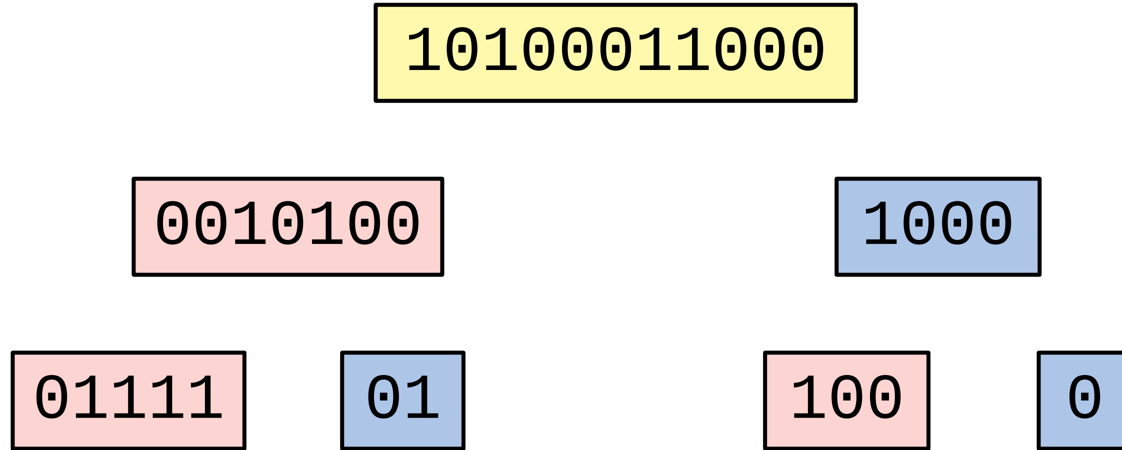
The Wavelet Matrix



The Wavelet Matrix



The Wavelet Matrix



group left children at the left, right children at the right

The Wavelet Matrix

10100011000

0010100 | 1000

01111 | 100 | 01 | 0

The Wavelet Matrix

Storage

10100011000

0010100	1000

01111100	010

The Wavelet Matrix
Storage

10100011000	→ 7 zeroes
0010100 1000	→ 8 zeroes
01111100 010	

The Wavelet Matrix

Storage

10100011000	→ 7 zeroes
0010100 | 1000	→ 8 zeroes
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- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select

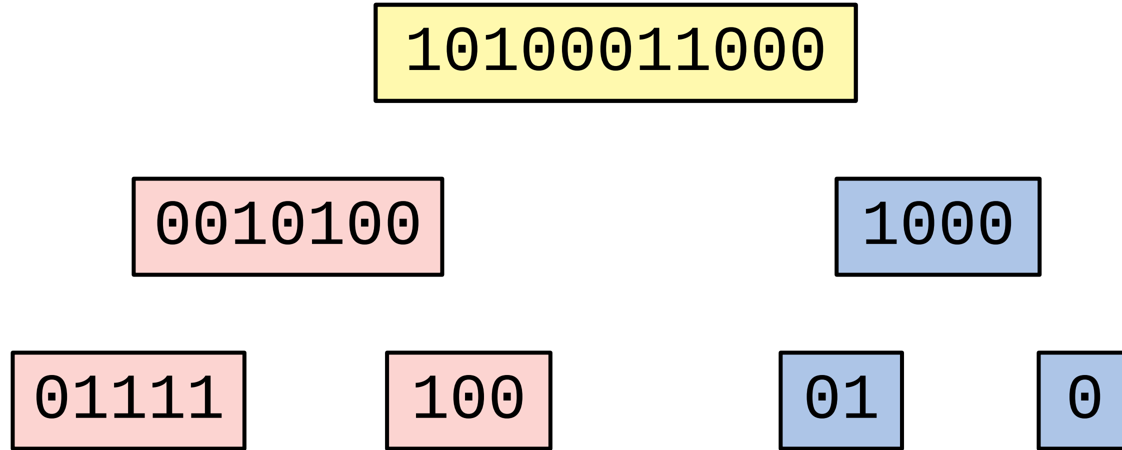
The Wavelet Matrix
Storage

```
10100011000   → 7 zeroes

0010100|1000  → 8 zeroes

01111100|010
```

- $(n+o(n)) \lceil lg\ \sigma \rceil$ bits for bit vectors + rank/select
- affordable $\lceil lg\ n \rceil \lceil lg\ \sigma \rceil$ bits for zeroes (boundaries)

The Wavelet Matrix

… as a tree



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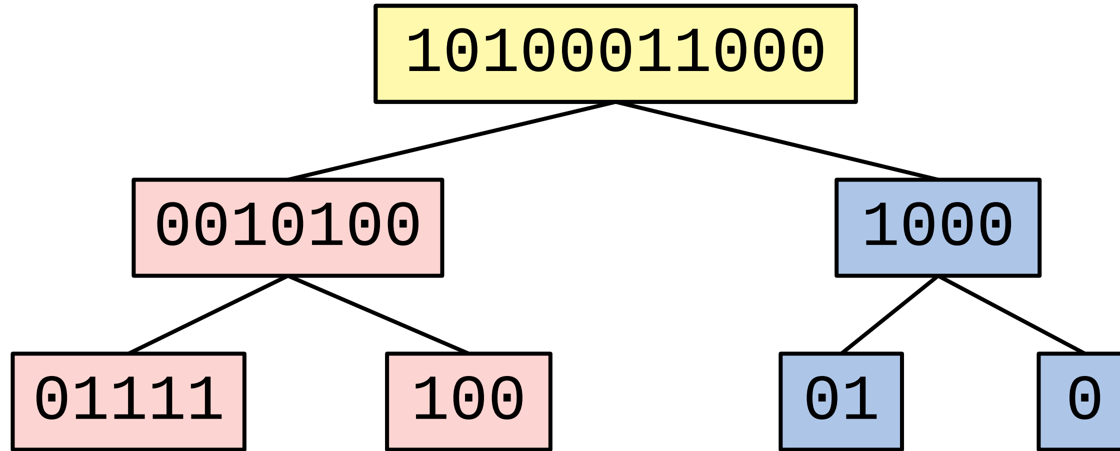

The Wavelet Matrix

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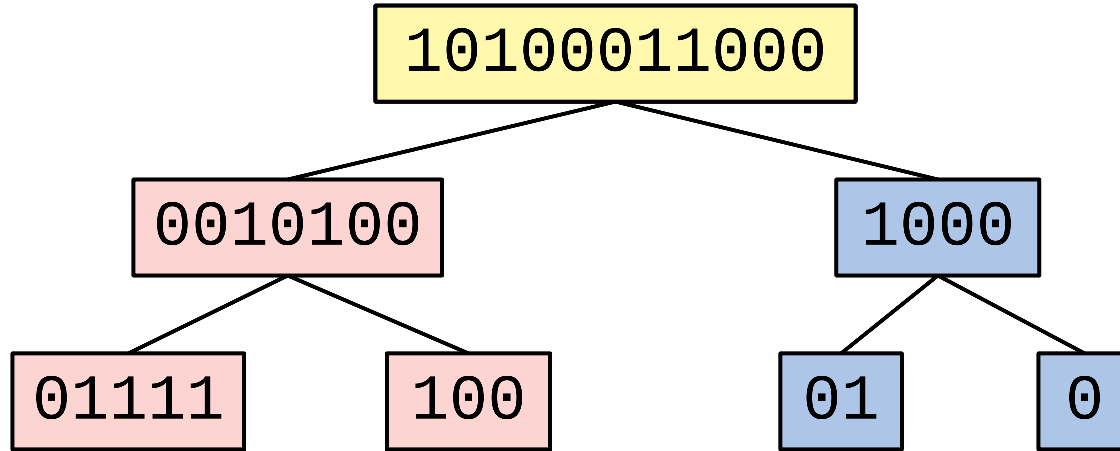
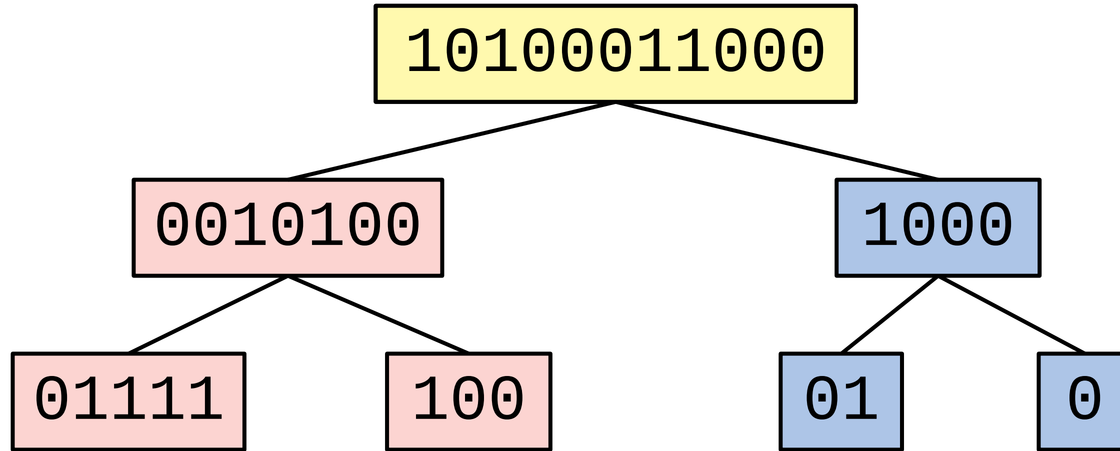


➢ will be useful for our cause

The Wavelet Matrix

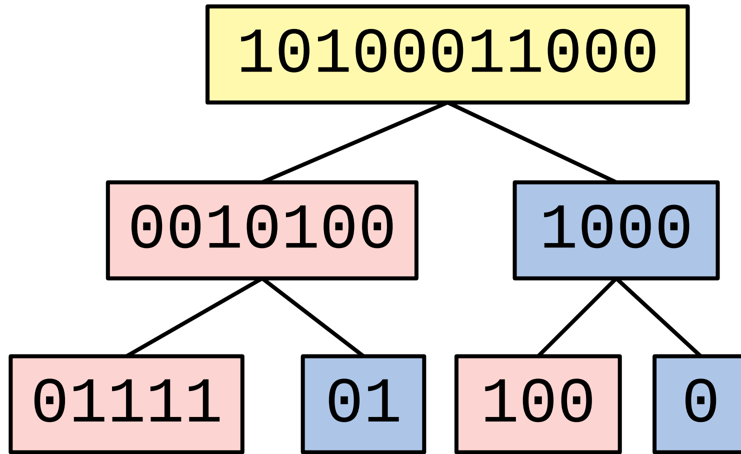
… as a tree

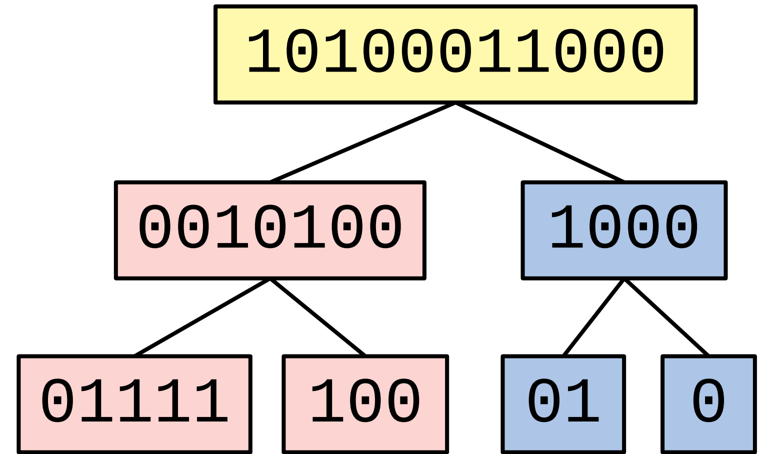


➢ will be useful for our cause

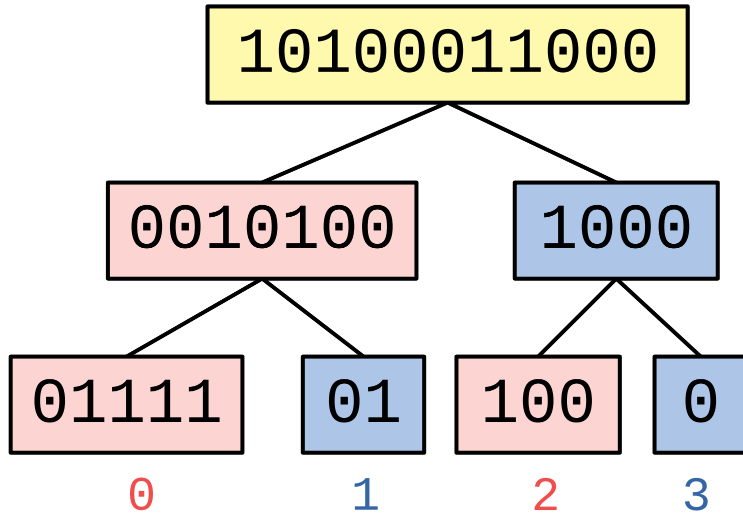
(but is not of practical relevance)

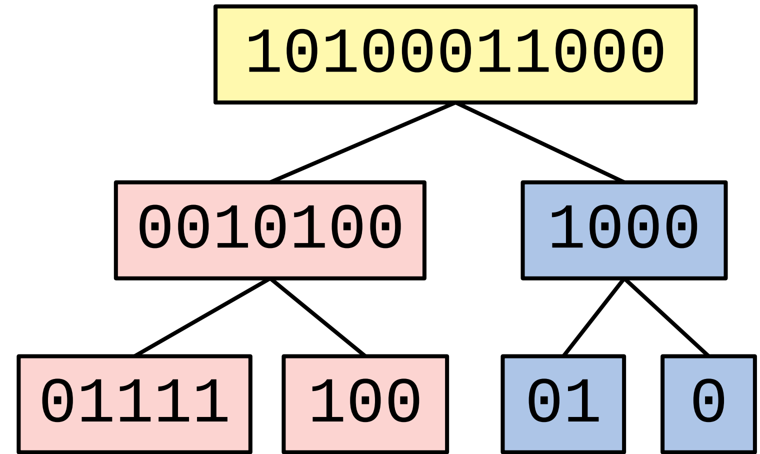
Wavelet Tree vs Wavelet Matrix





Wavelet Tree vs Wavelet Matrix





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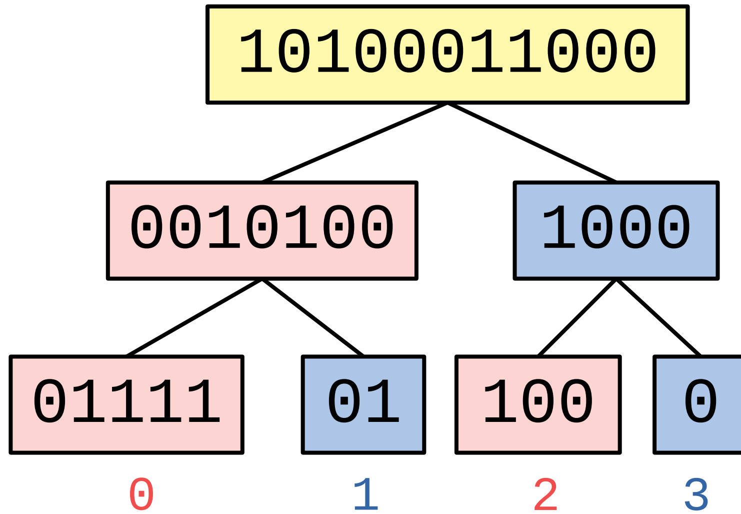

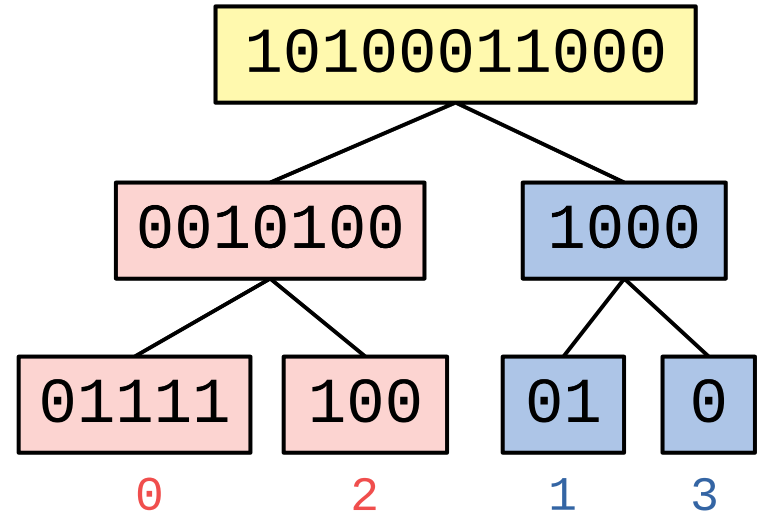

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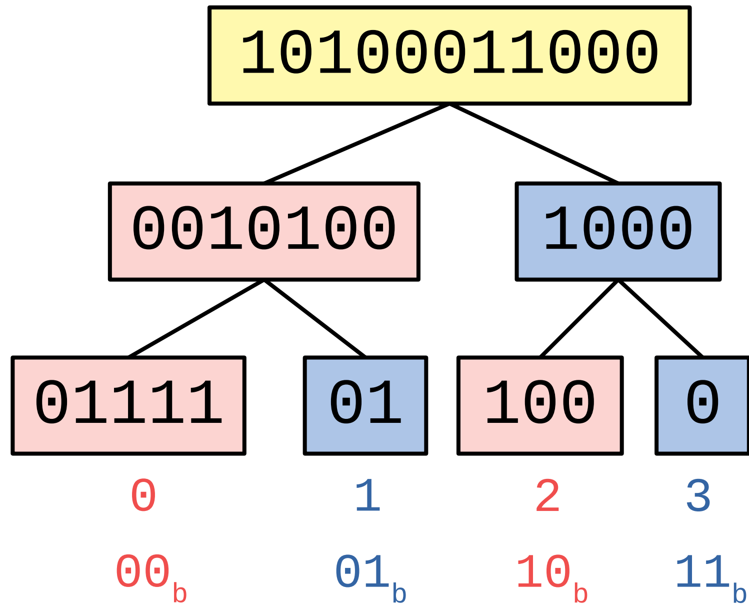
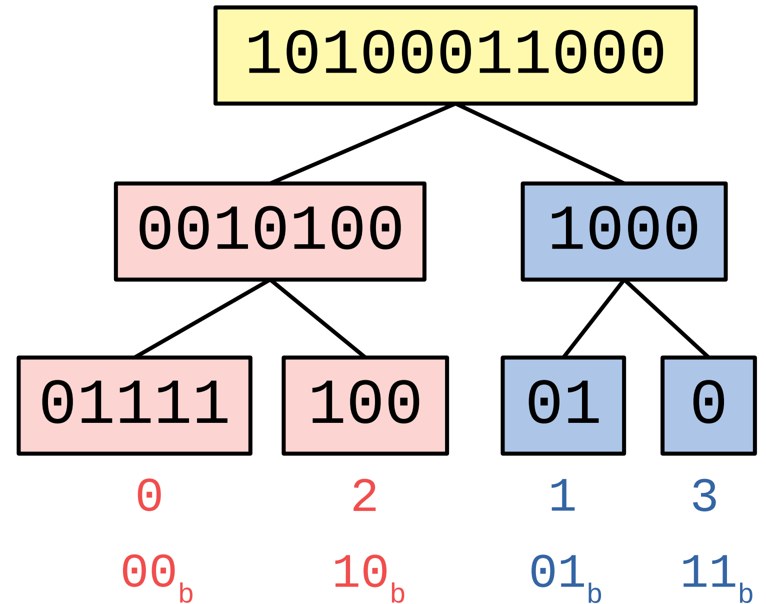
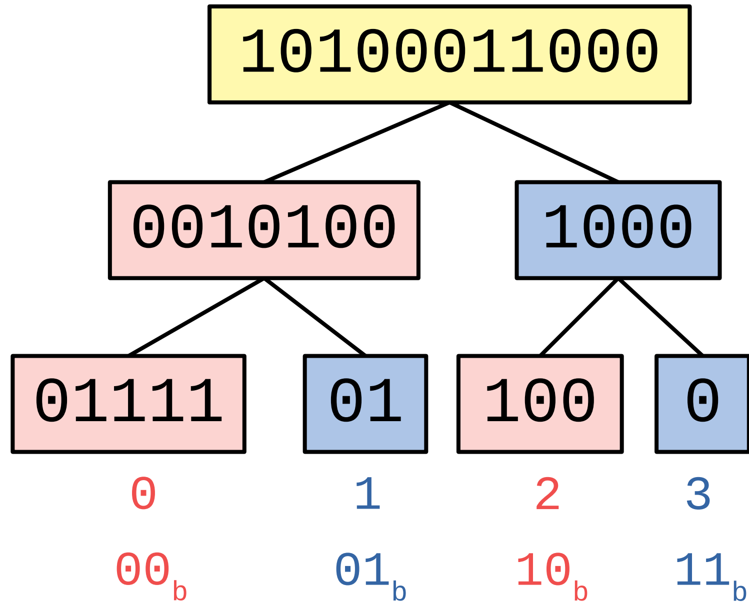
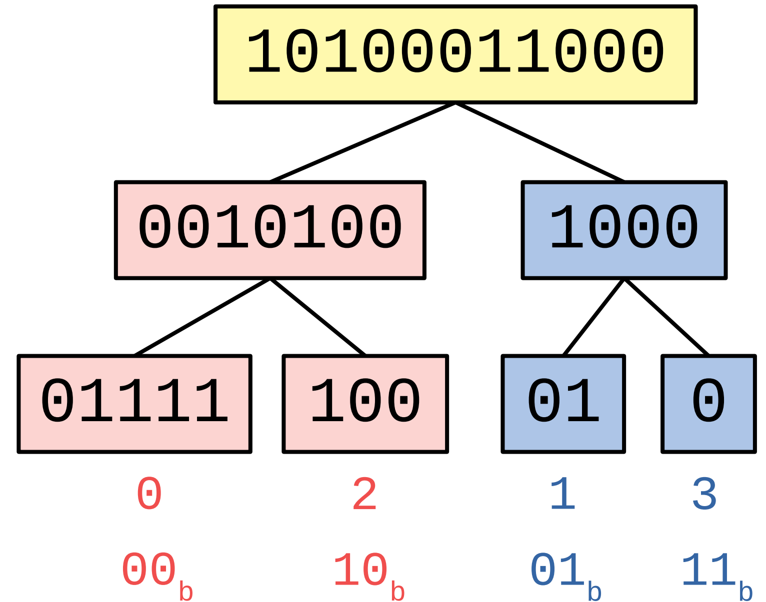




Wavelet Tree vs Wavelet Matrix





➢ bit-reversal permutation!

Translation Problem

Given: WTCA, level ℓ, position $i < n$

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WTCA about to write $\boldsymbol{i}$-th bit on level ℓ of the wavelet tree.

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*WTCA writes bit at position j instead and becomes W**M**CA.*

Translation Problem

Given: WTCA, level ℓ, position $i < n$

WTCA about to write i-th bit on level ℓ of the wavelet tree.

Find (efficiently): Position j < n

*WTCA writes bit at position j instead and becomes W**M**CA.*

- solved by Fischer, Kurpicz & Löbel [ALENEX, 2018]
  (DS of size $(n+\sigma)(1+o(1)) + (\sigma+2)\lceil lg\ n \rceil$ bits)

Translation Problem

The Inverse

Given: WMCA, level ℓ, position $j < n$

Translation Problem
The Inverse

Given: WMCA, level ℓ, position $j < n$

WMCA about to write j-th bit on level ℓ of the wavelet matrix.

Find (efficiently): Position i < n

*WMCA writes bit at position i instead and becomes W**T**CA.*

Translation Problem

The Inverse

Given: WMCA, level ℓ, position $j < n$

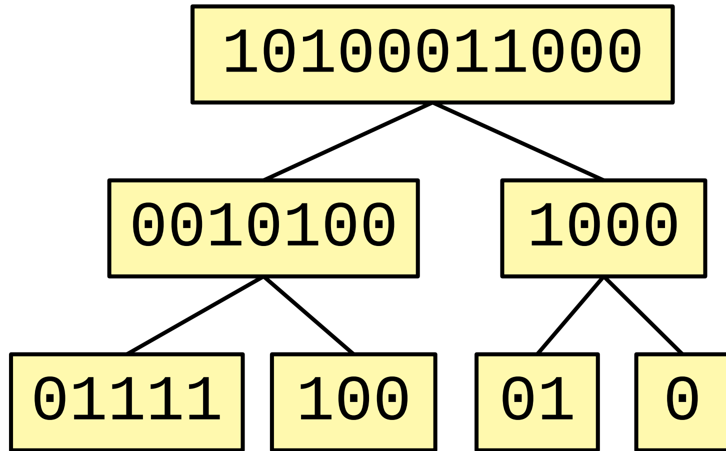
WMCA about to write j-th bit on level ℓ of the wavelet matrix.

Find (efficiently): Position i < n

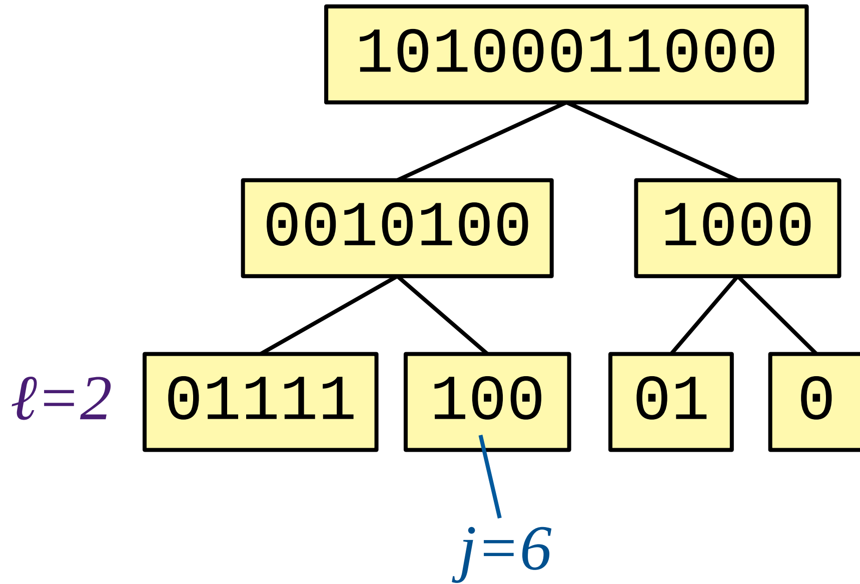
*WMCA writes bit at position i instead and becomes W**T**CA.*

➢ **this work**

Strategy

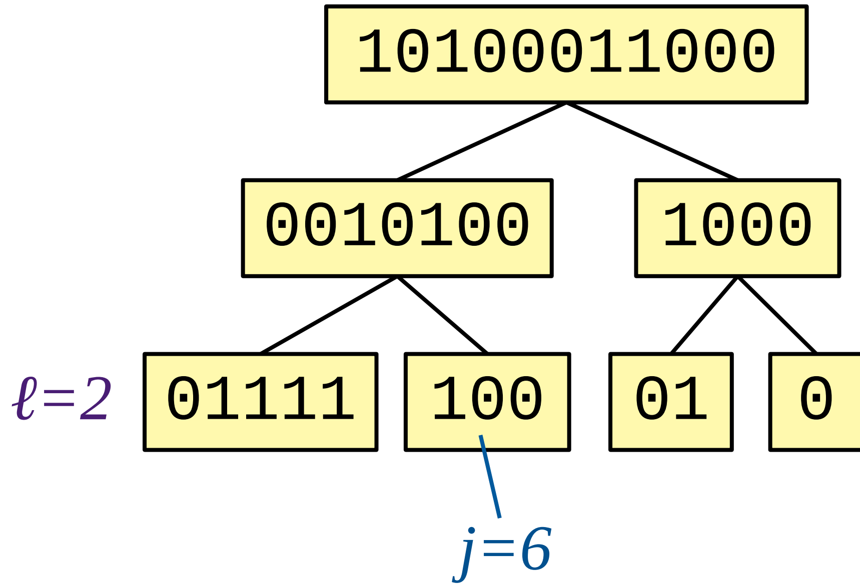


Strategy

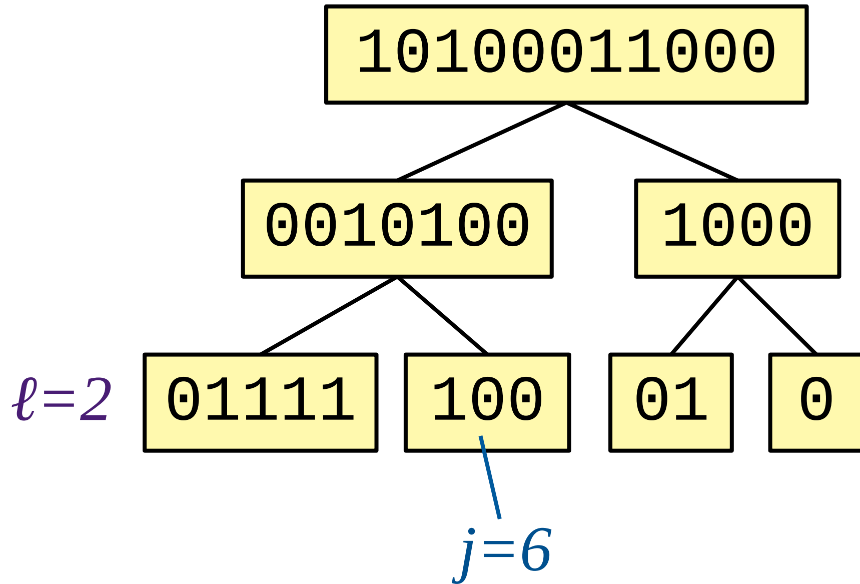


Strategy

Locate:



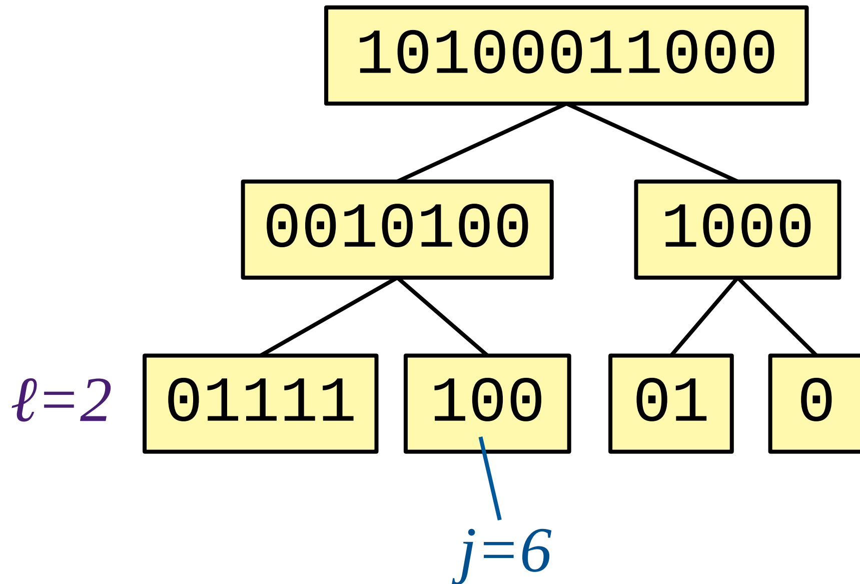
Strategy



Locate:

- The node that contains pos. i or j

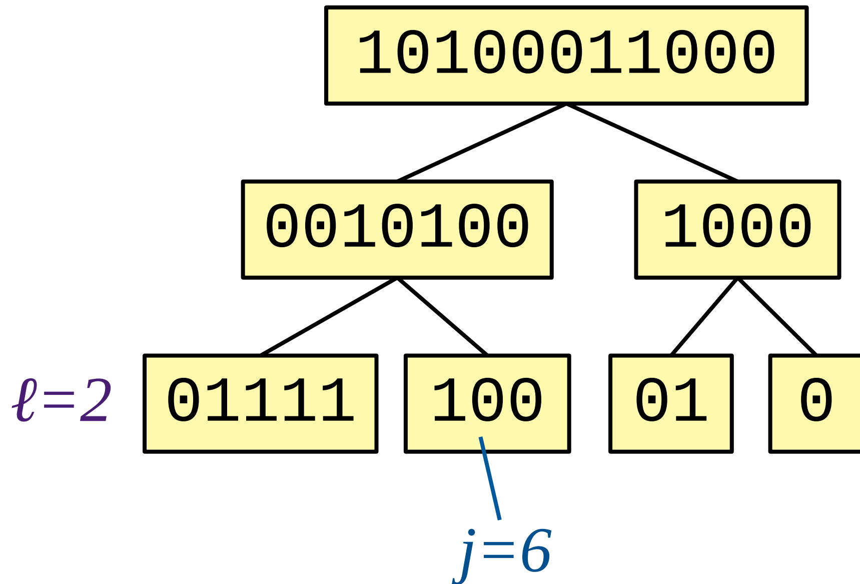
Strategy



Locate:

- The node that contains pos. i or j
- The position of its first bit

Strategy

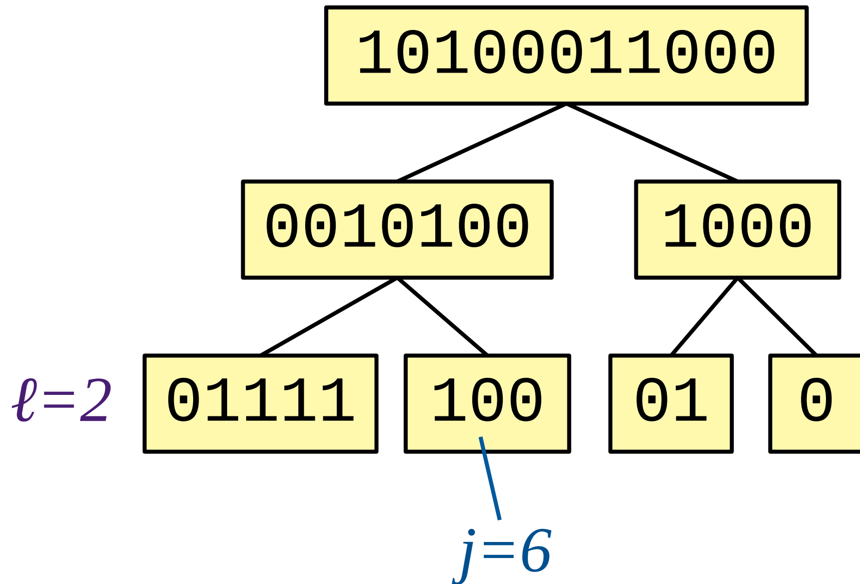


Locate:

- The node that contains pos. i or j
- The position of its first bit

Then:

Strategy



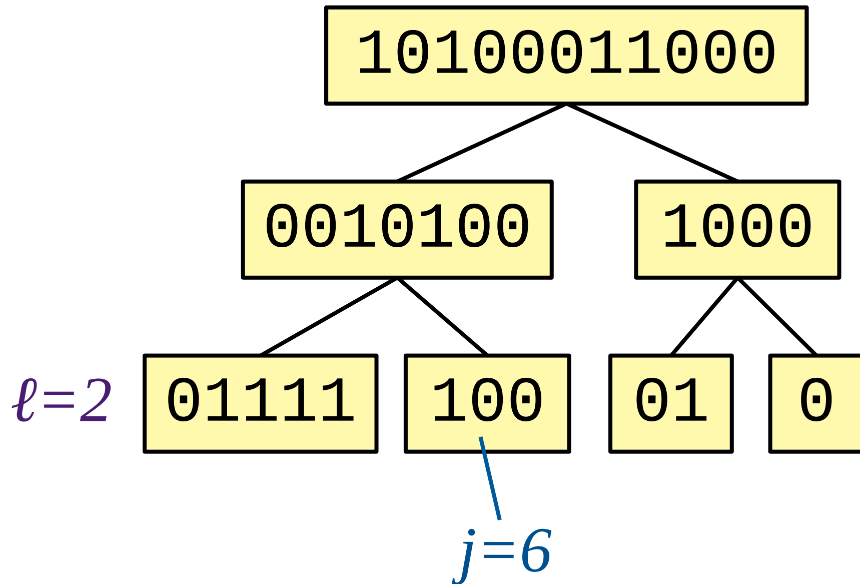
Locate:

- The node that contains pos. i or j
- The position of its first bit

Then:

- Use bit reversal to find counterpart node

Strategy



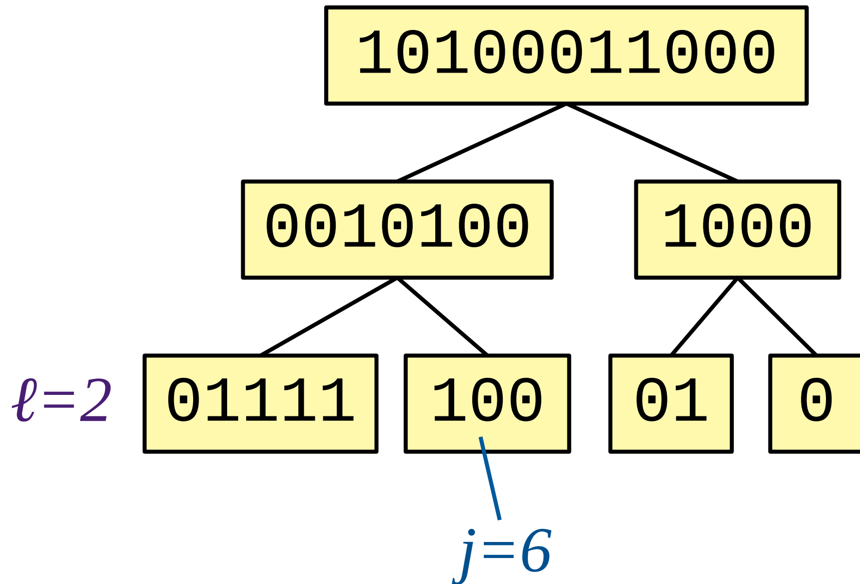
Locate:

- The node that contains pos. i or j
- The position of its first bit

Then:

- Use bit reversal to find counterpart node
- Find the node's first bit in counterpart

Strategy



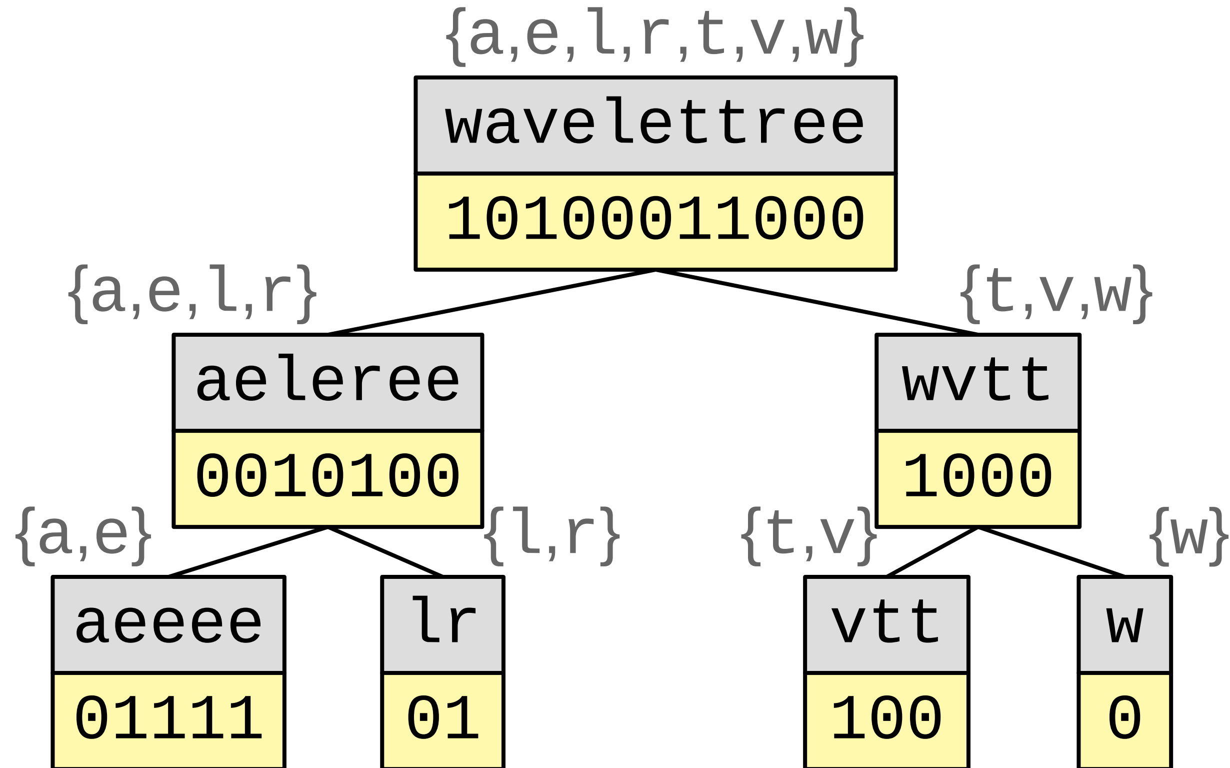
Locate:

- The node that contains pos. i or j
- The position of its first bit

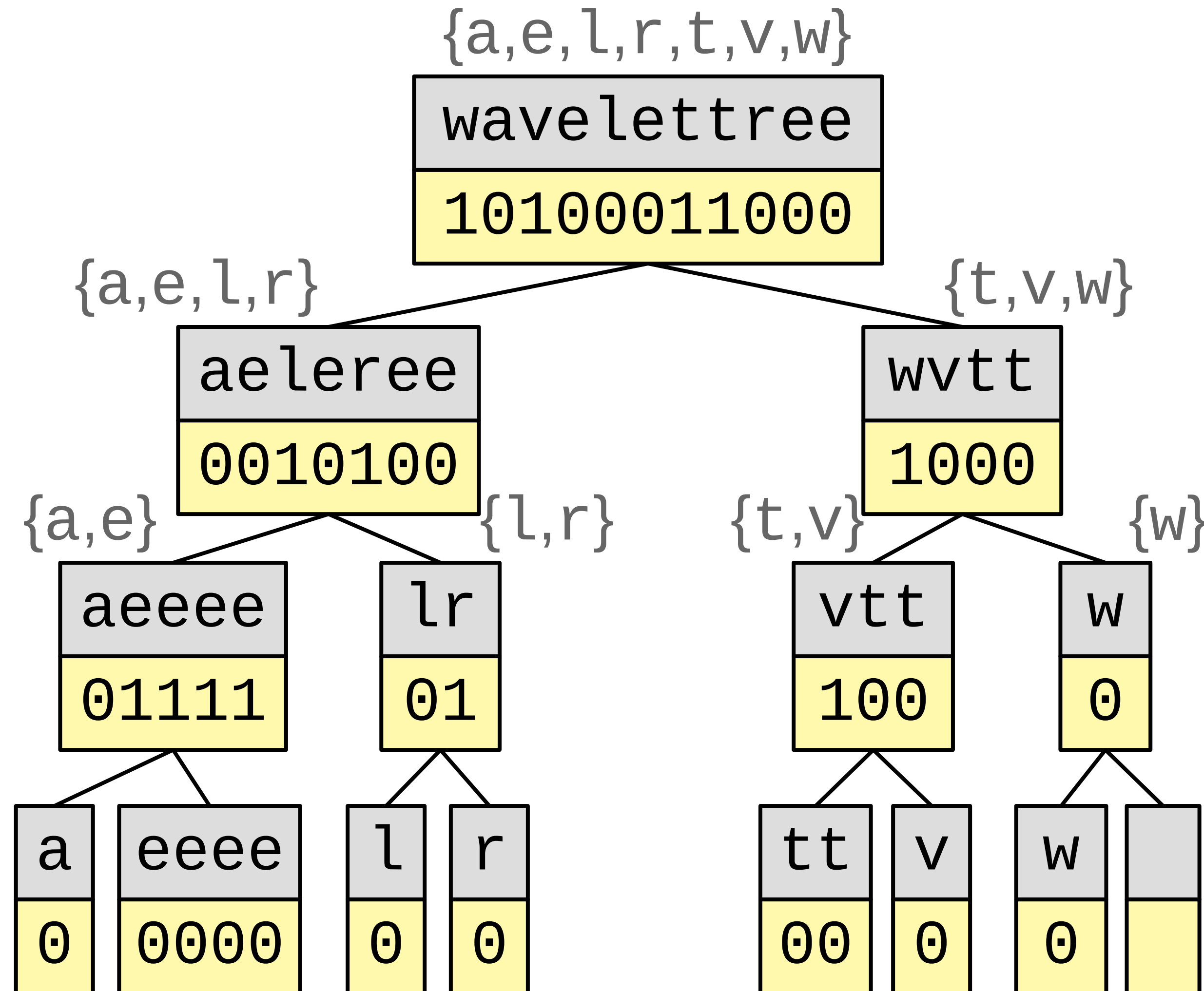
Then:

- Use bit reversal to find counterpart node
- Find the node's first bit in counterpart
- Add offset

Virtual Bottom Level



Virtual Bottom Level

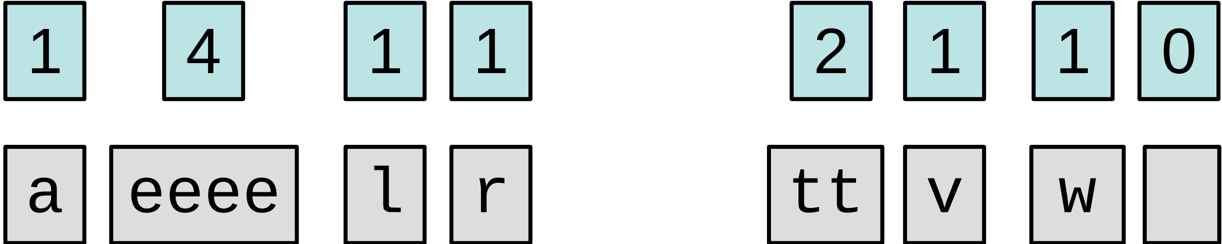


Virtual Bottom Level

wavelettree

Virtual Bottom Level

wavelettree



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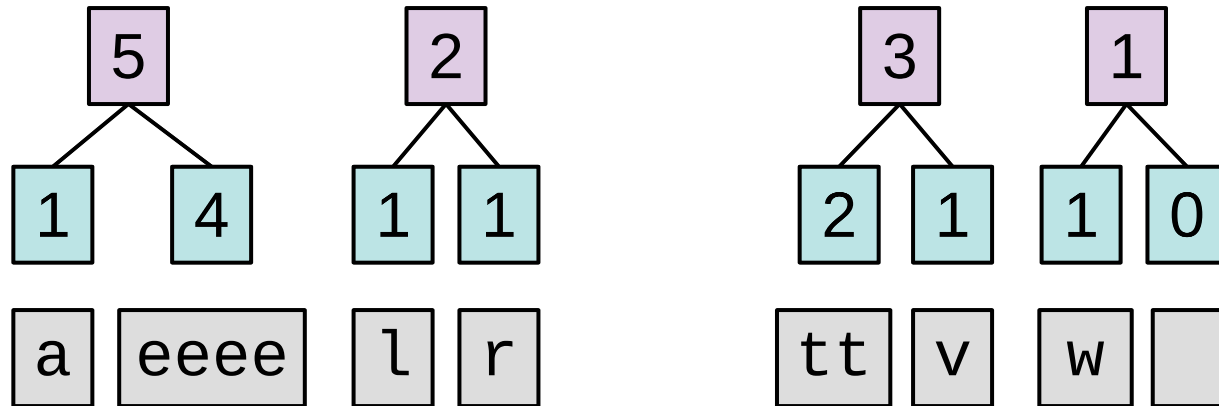

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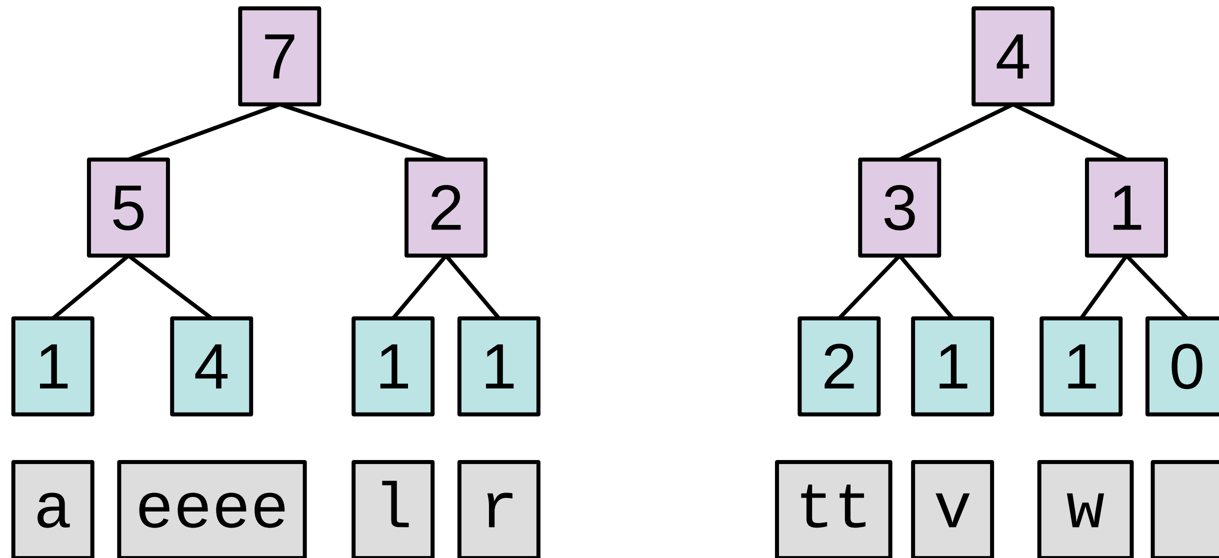

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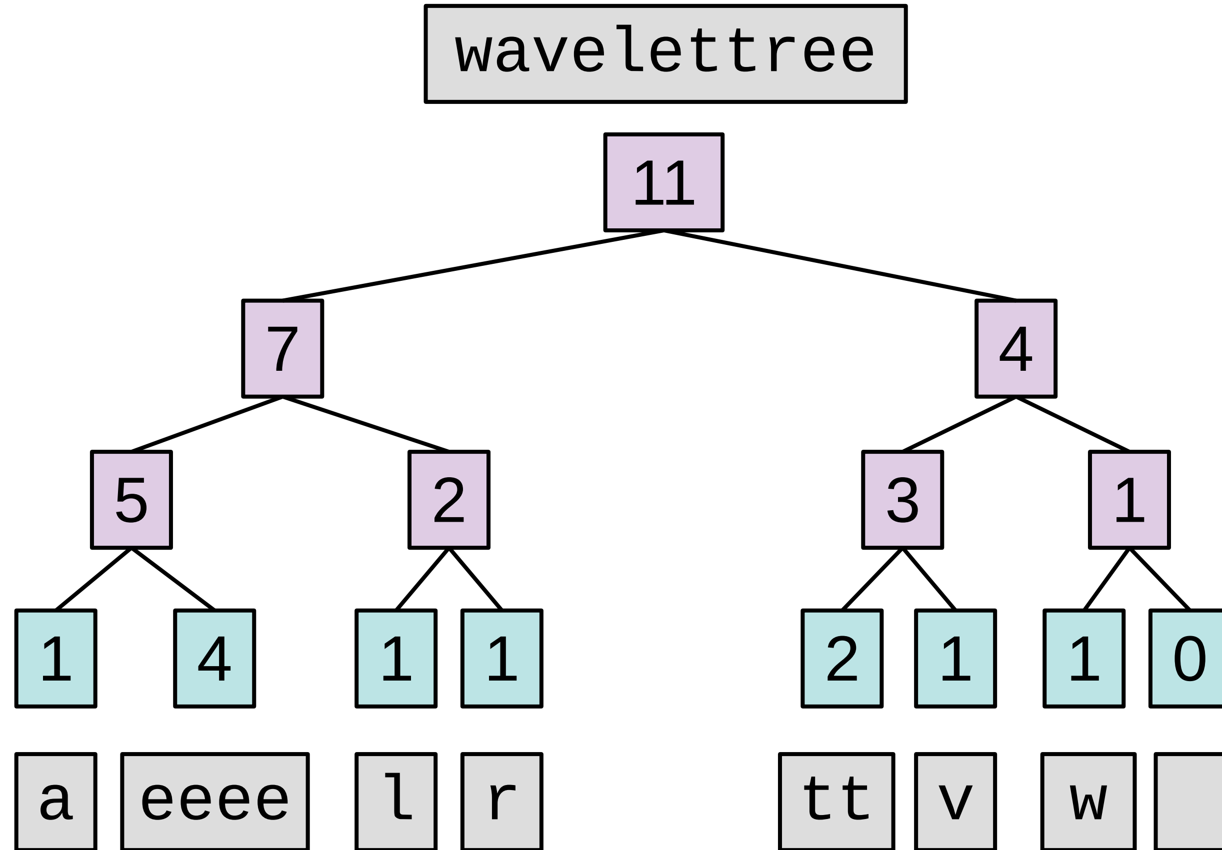
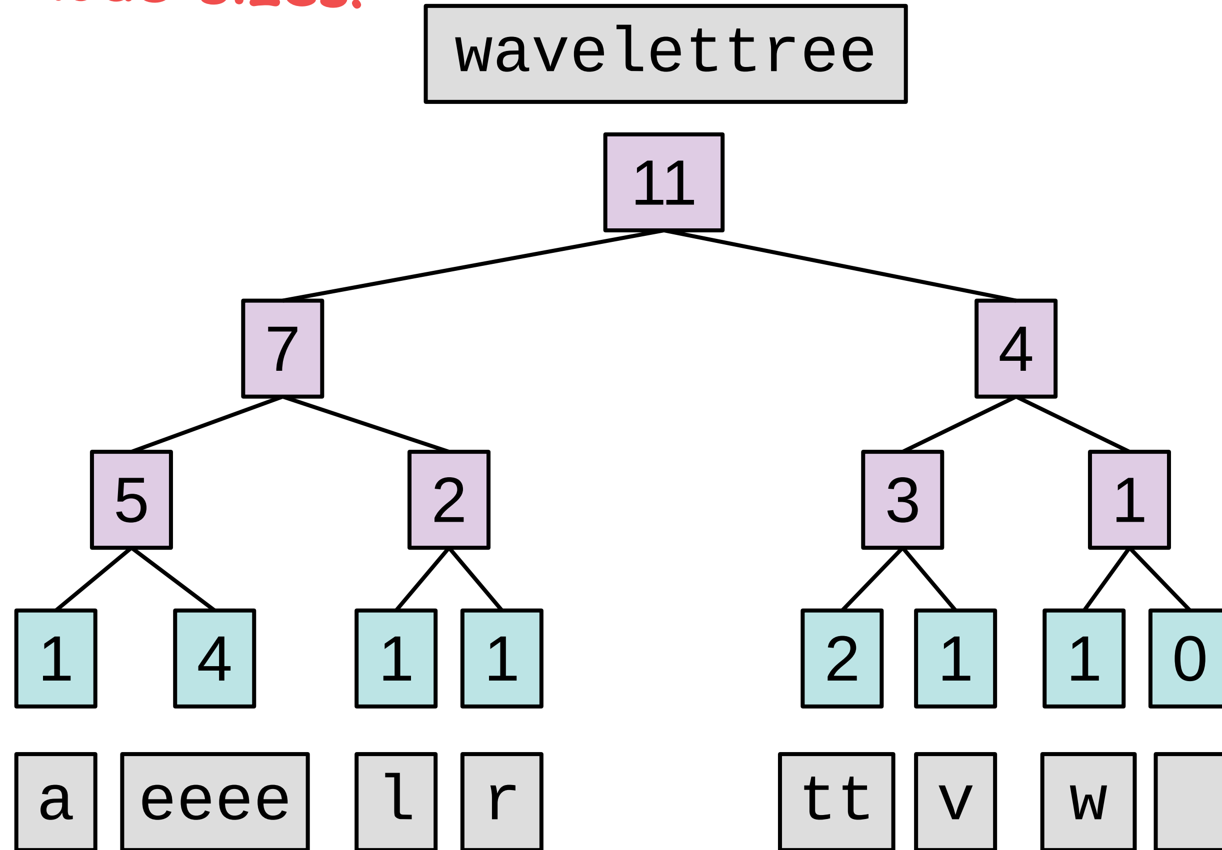


Virtual Bottom Level

encodes all node sizes!



The C Array

c	a	e	l	r	t	v	w	
	0	1	2	3	4	5	6	7

The C Array

c	a 0	e 1	l 2	r 3	t 4	v 5	w 6	 7
$H[c]$	1	4	1	1	2	1	1	

The C Array

c	a 0	e 1	l 2	r 3	t 4	v 5	w 6	 7
$H[c]$	1	4	1	1	2	1	1	
$\mathbf{C[c]}$	0	1	5	6	7	9	10	11

The C Array

c	a 0	e 1	l 2	r 3	t 4	v 5	w 6	 7
$H[c]$	1	4	1	1	2	1	1	
$C[c]$	0	1	5	6	7	9	10	11

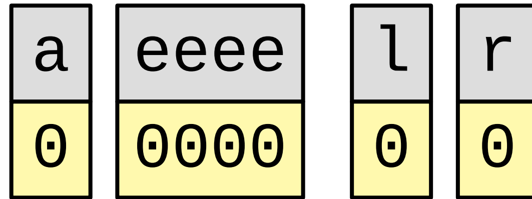
$$H[c] = C[c+1] - C[c]$$

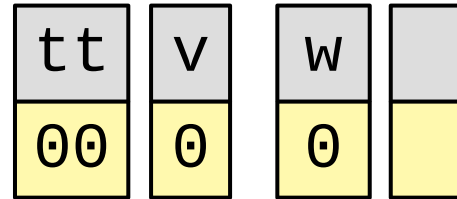
The C Array

c	a 0	e 1	l 2	r 3	t 4	v 5	w 6	7
~~$H[c]$~~	~~1~~	~~4~~	~~1~~	~~1~~	~~2~~	~~1~~	~~1~~	
$C[c]$	0	1	5	6	7	9	10	11

$$H[c] = C[c+1] - C[c]$$

Locating in the WT
on the virtual bottom level





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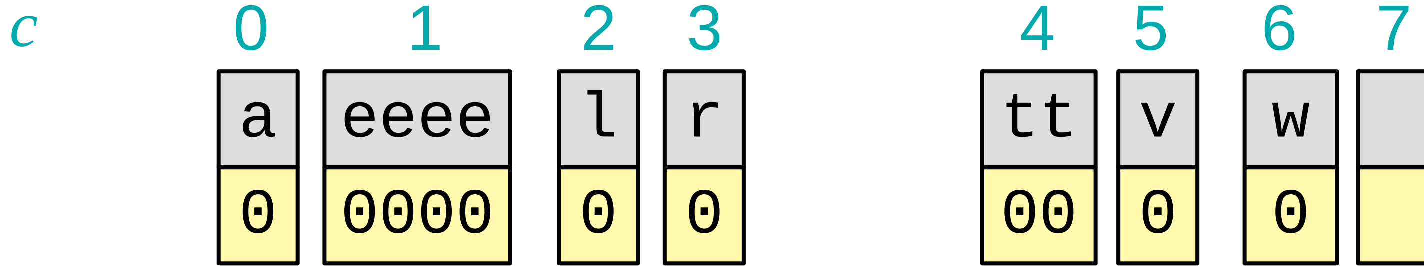
on the virtual bottom level



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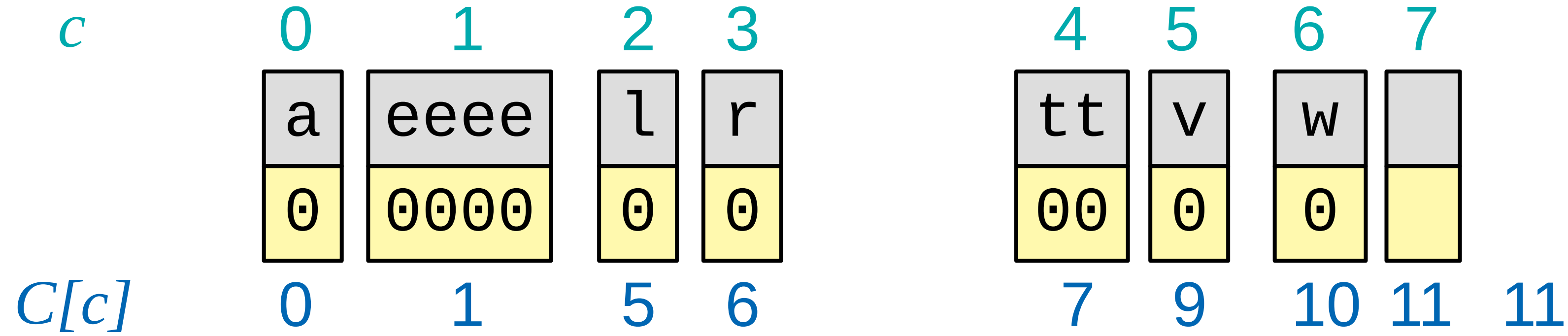
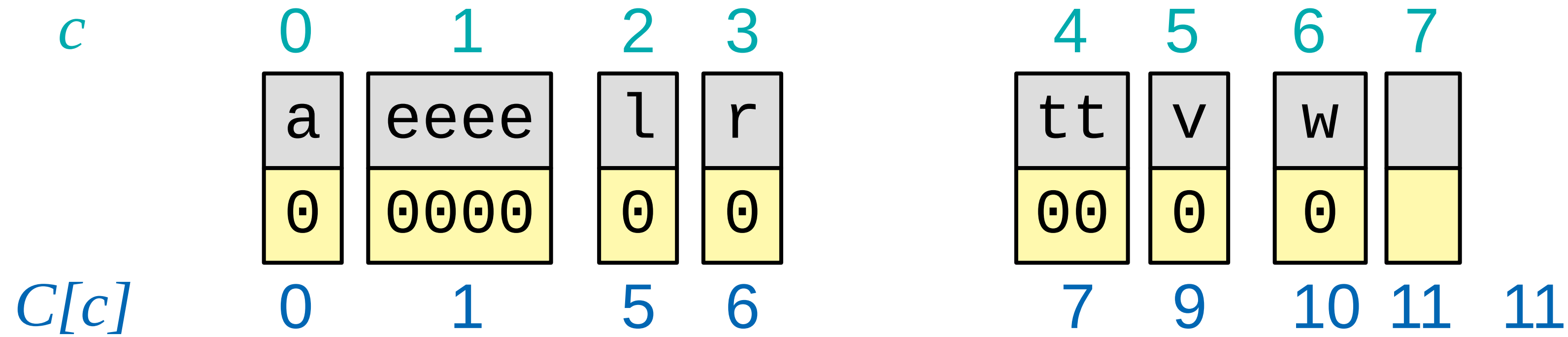
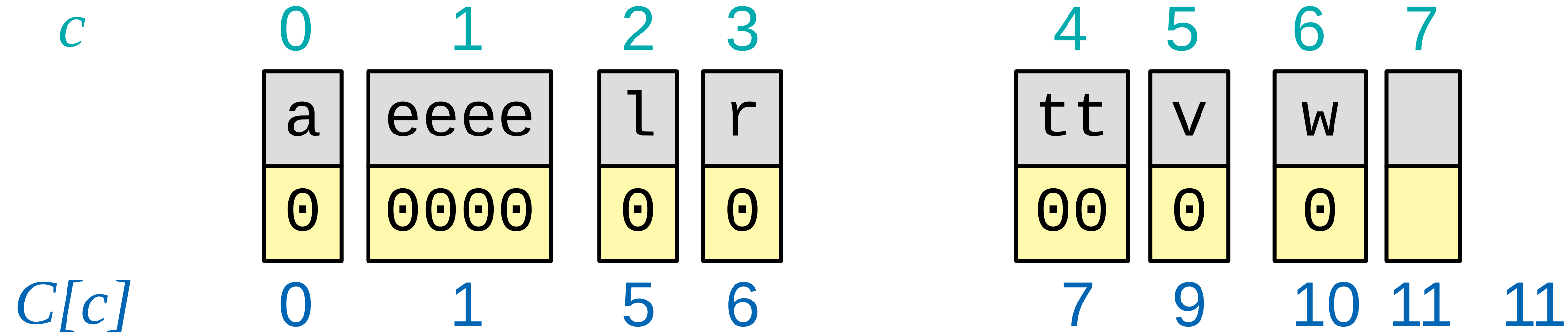


Locating in the WT
on the virtual bottom level



Given position ***i***:

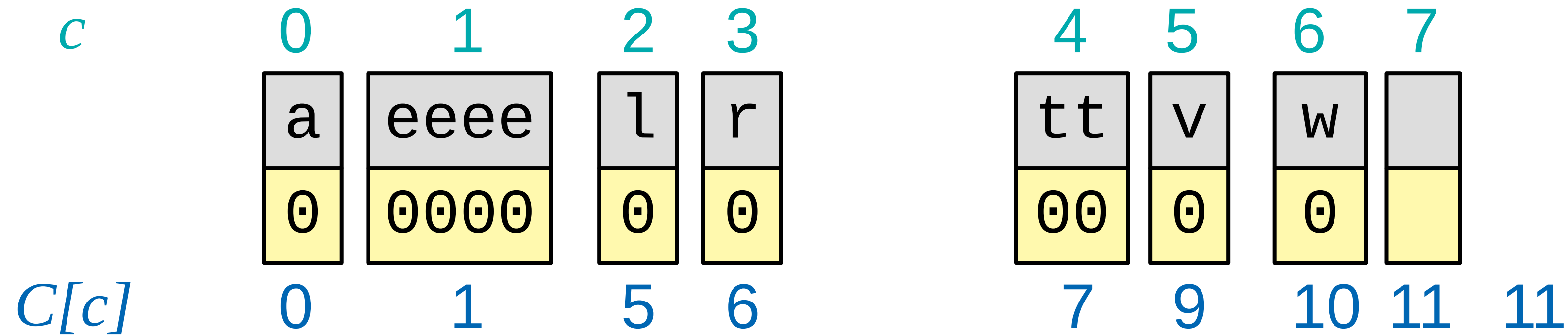
Locating in the WT
on the virtual bottom level



Given position i:

- Find $v = min\{c \mid C[c] > i\} - 1$

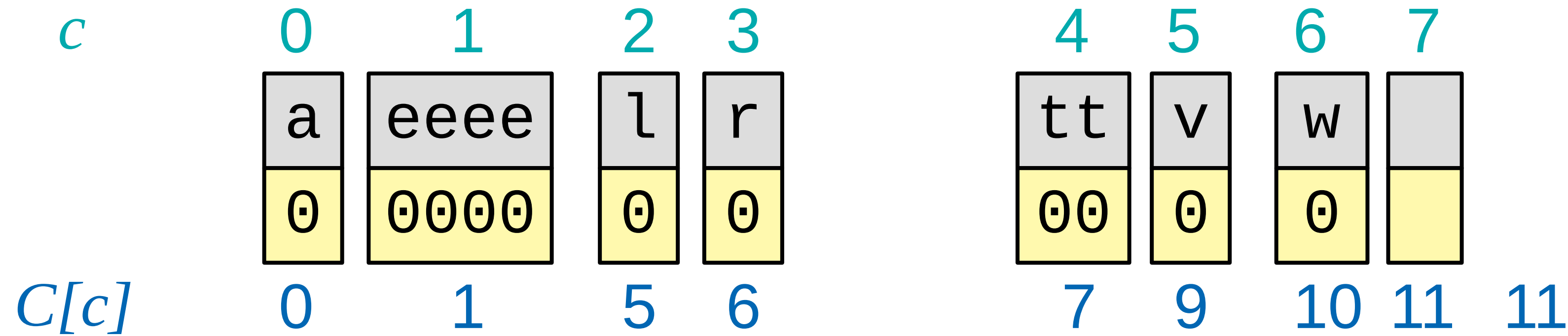
Locating in the WT
on the virtual bottom level



Given position i:

- Find $v = min\{c \mid C[c] > i\} - 1$
- v's first bit is located at $C[v]$

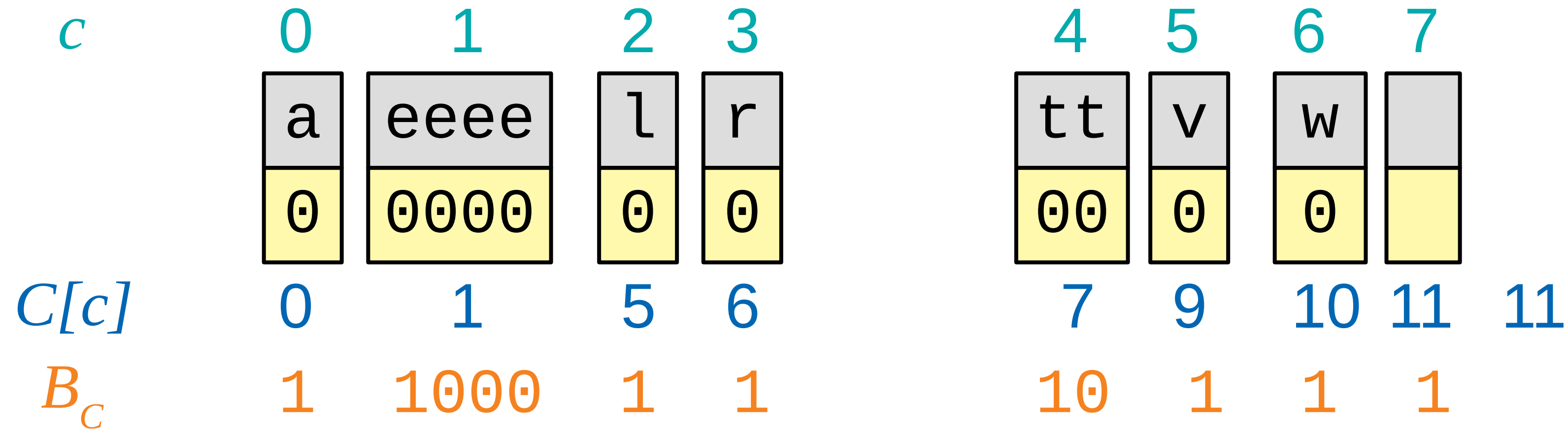
Locating in the WT
on the virtual bottom level



Given position i:

- Find $v = min\{c \mid C[c] > i\} - 1$ in constant time!
- v's first bit is located at $C[v]$

Locating in the WT
on the virtual bottom level



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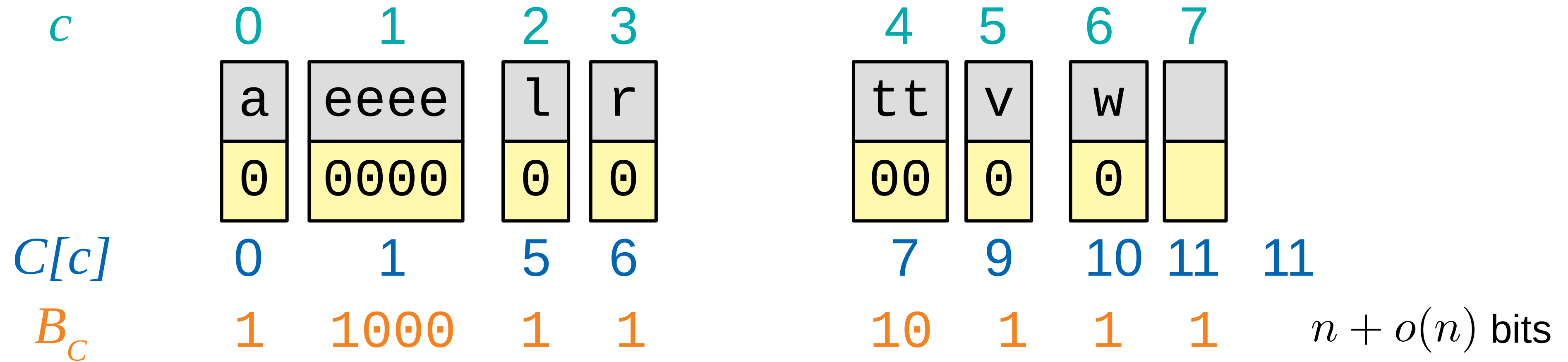
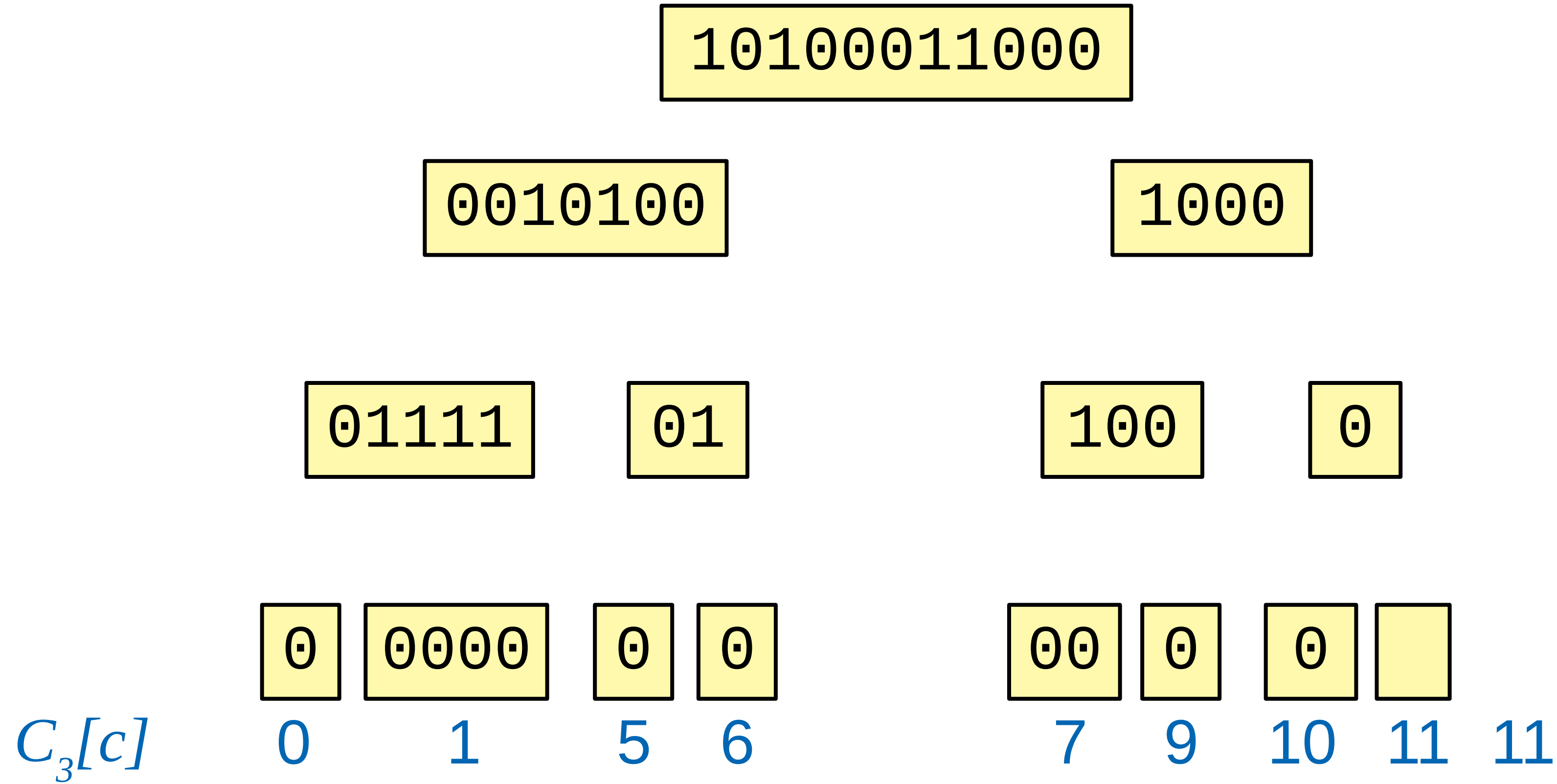


Given position ***i***:

- Find ***v*** = ***min{c* | *C[c]* > *i}* - 1** in constant time!
- ***v***'s first bit is located at ***C[v]***

Locating in the WT
on any level



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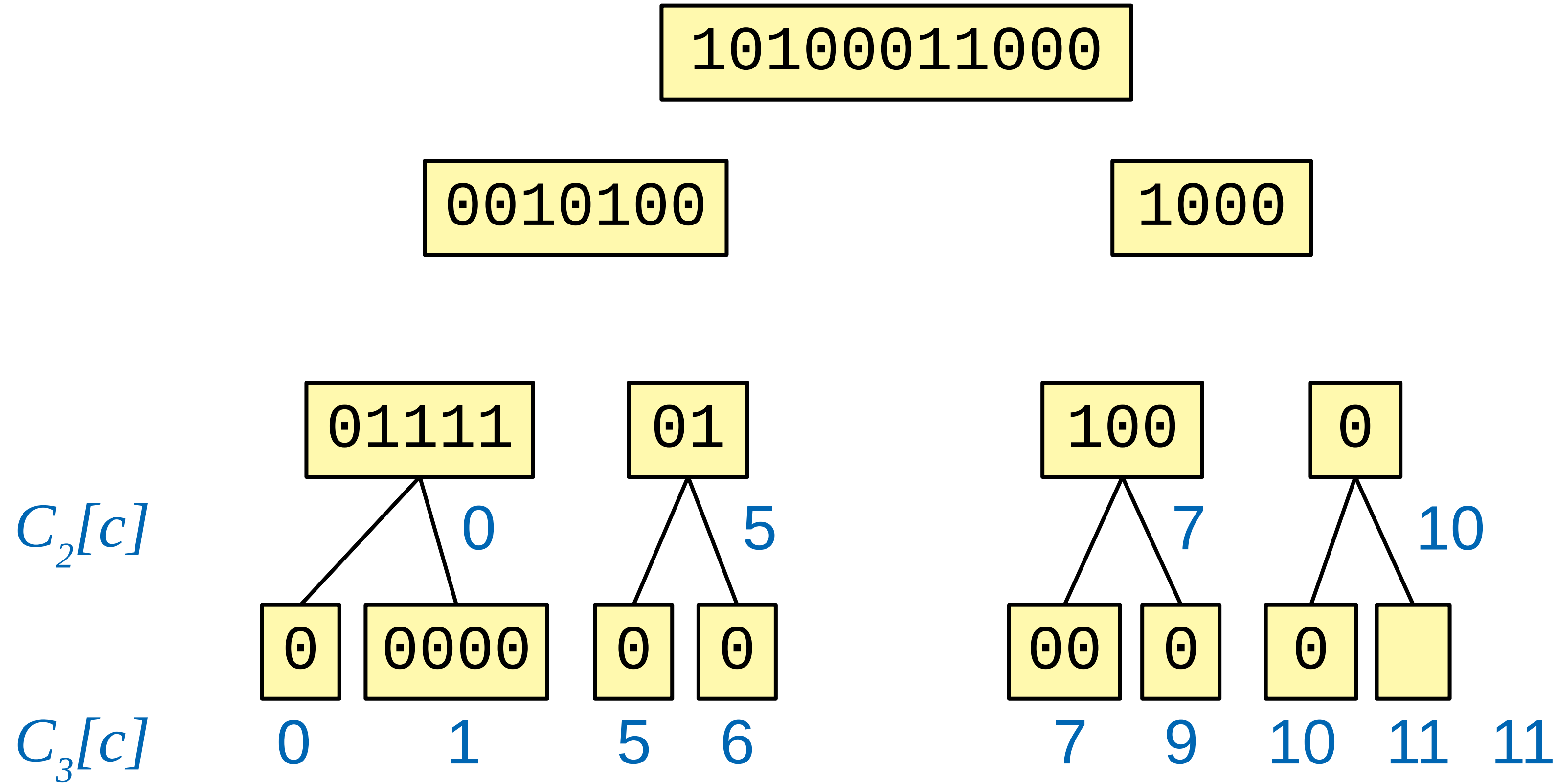

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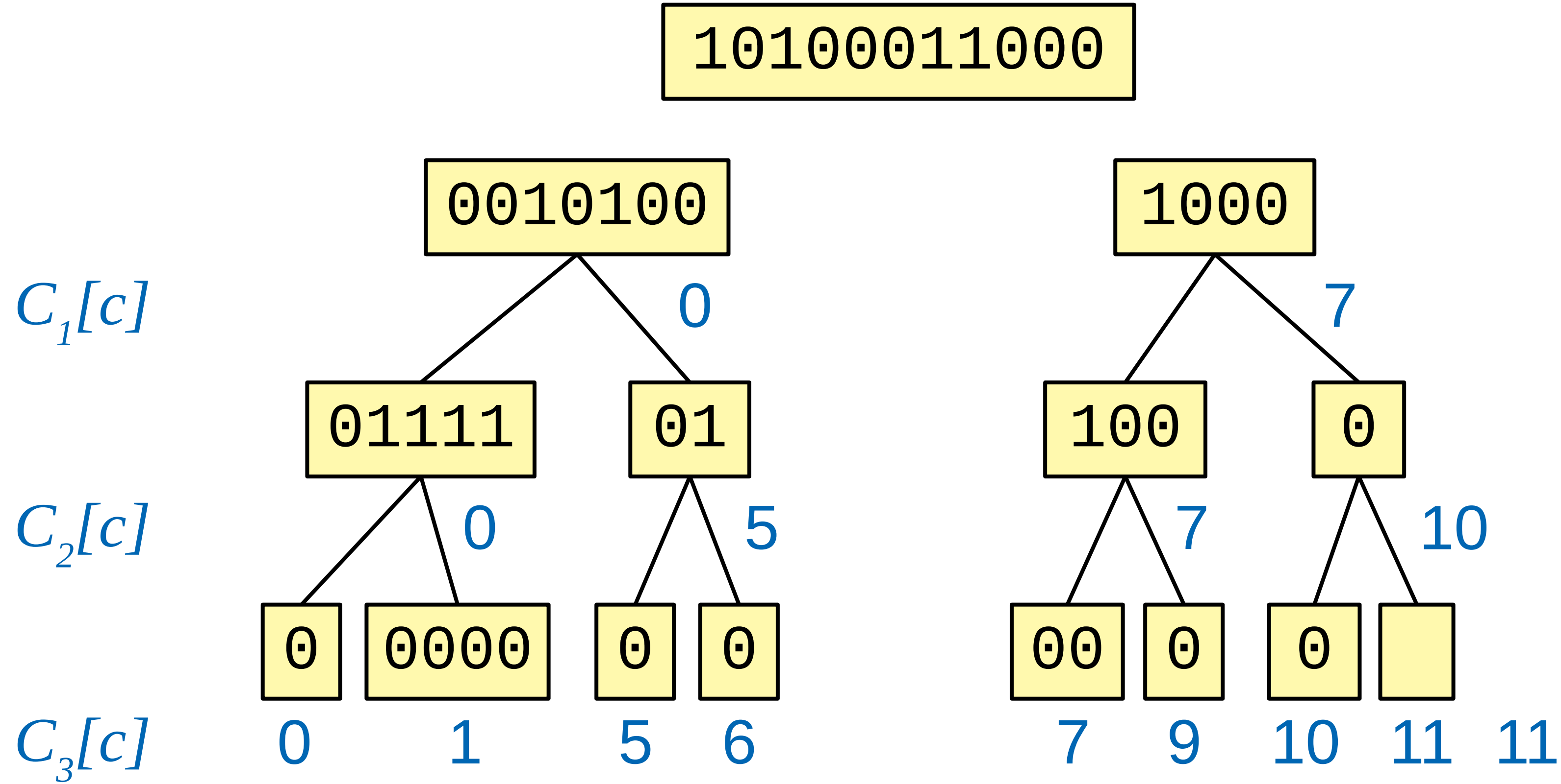
on any level



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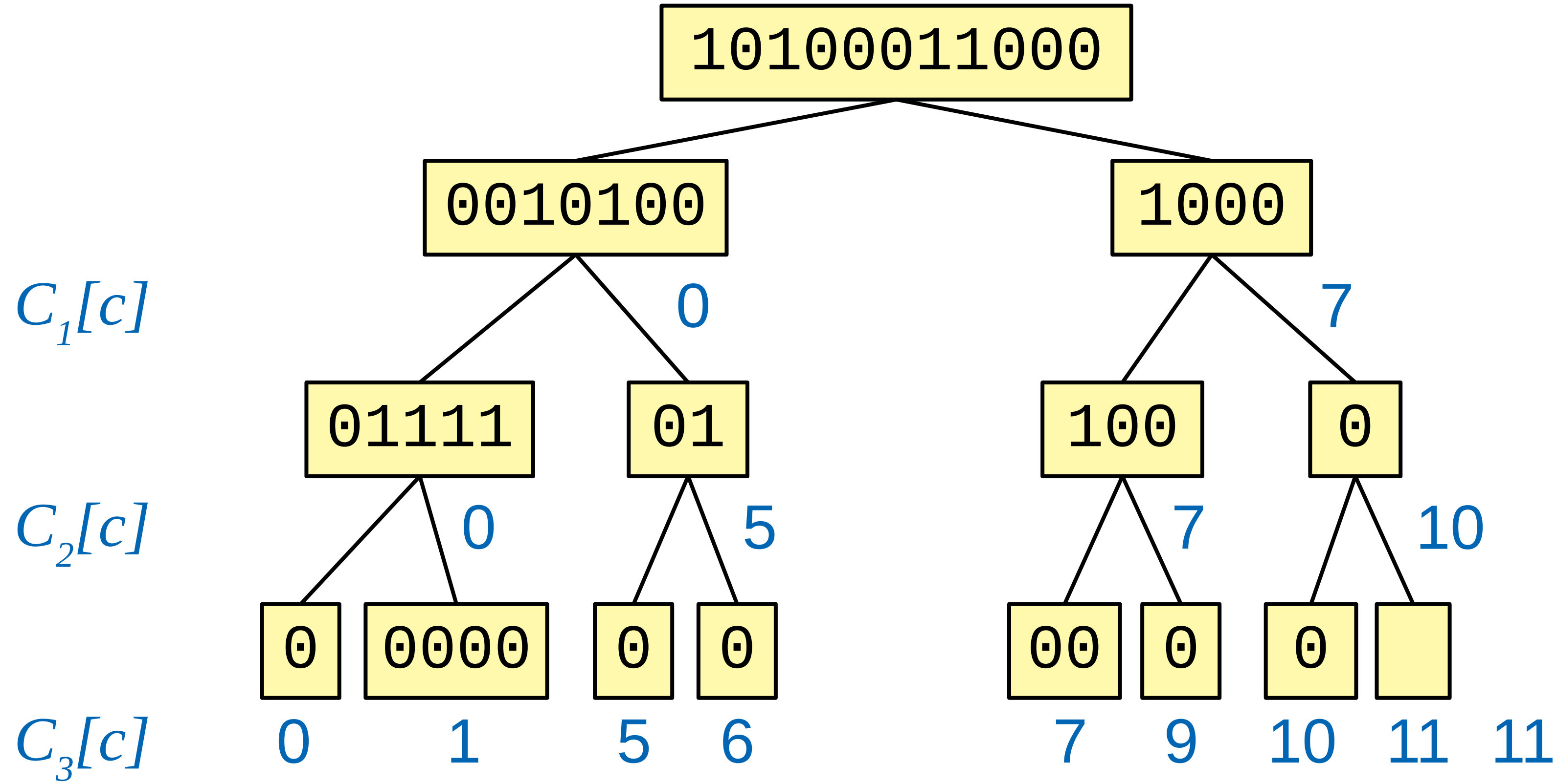

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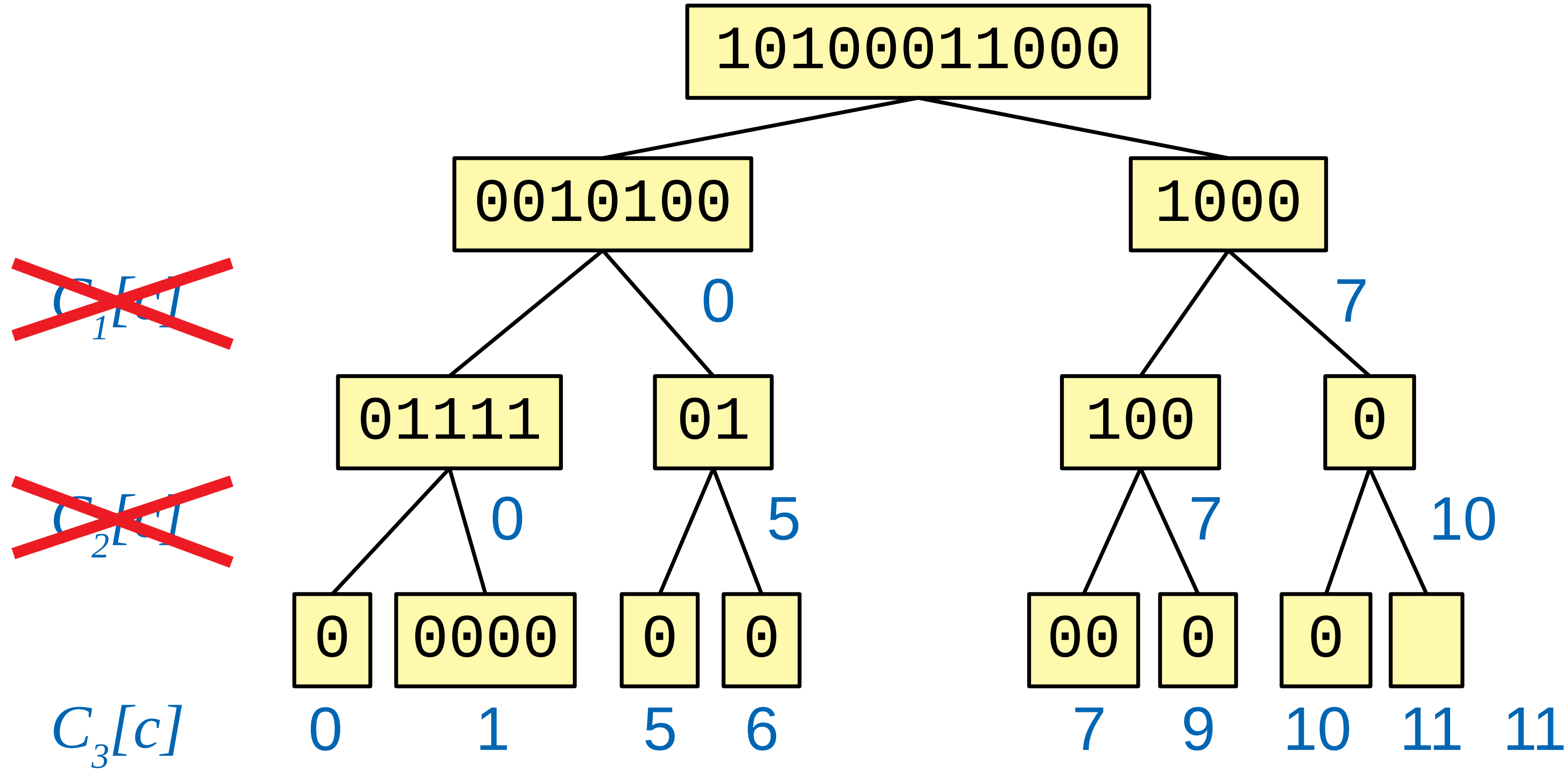
on any level



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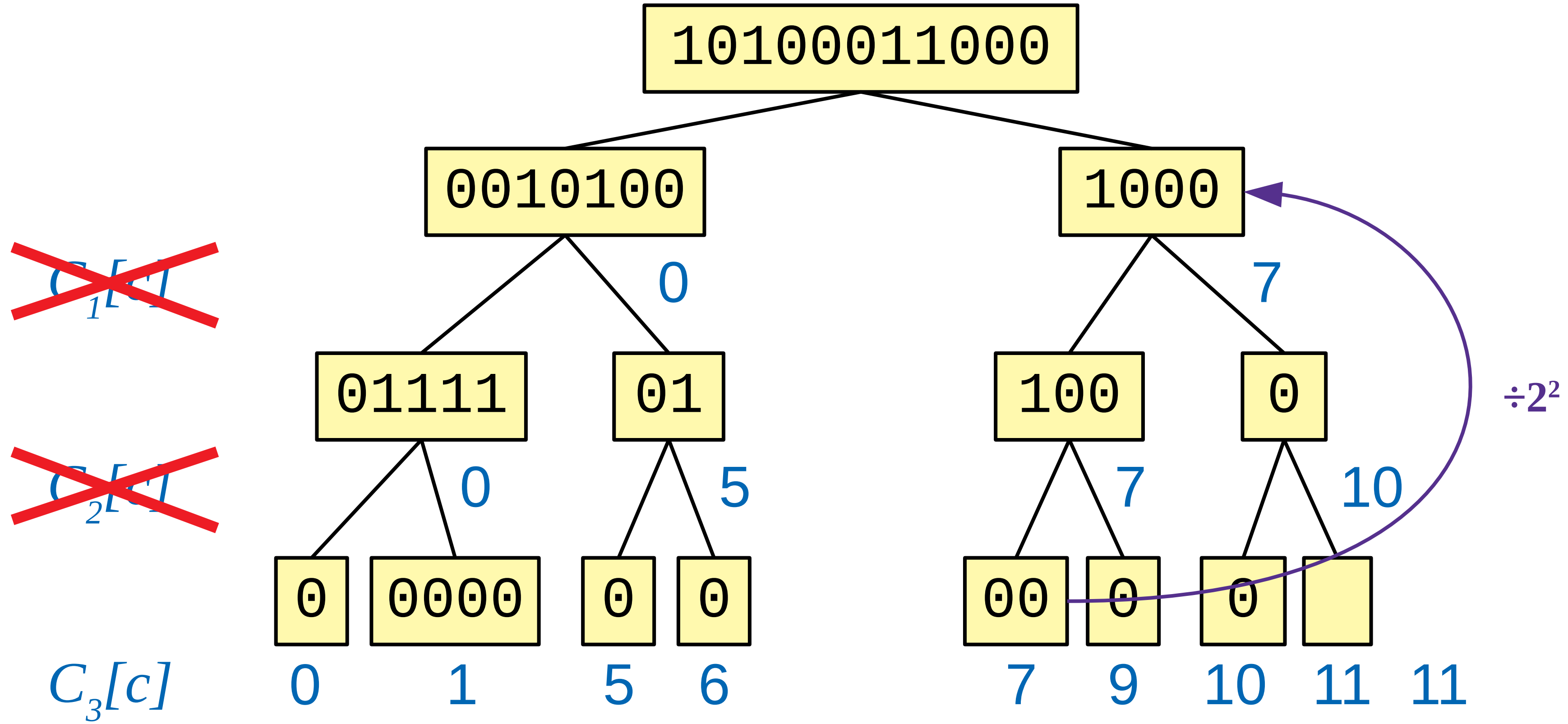

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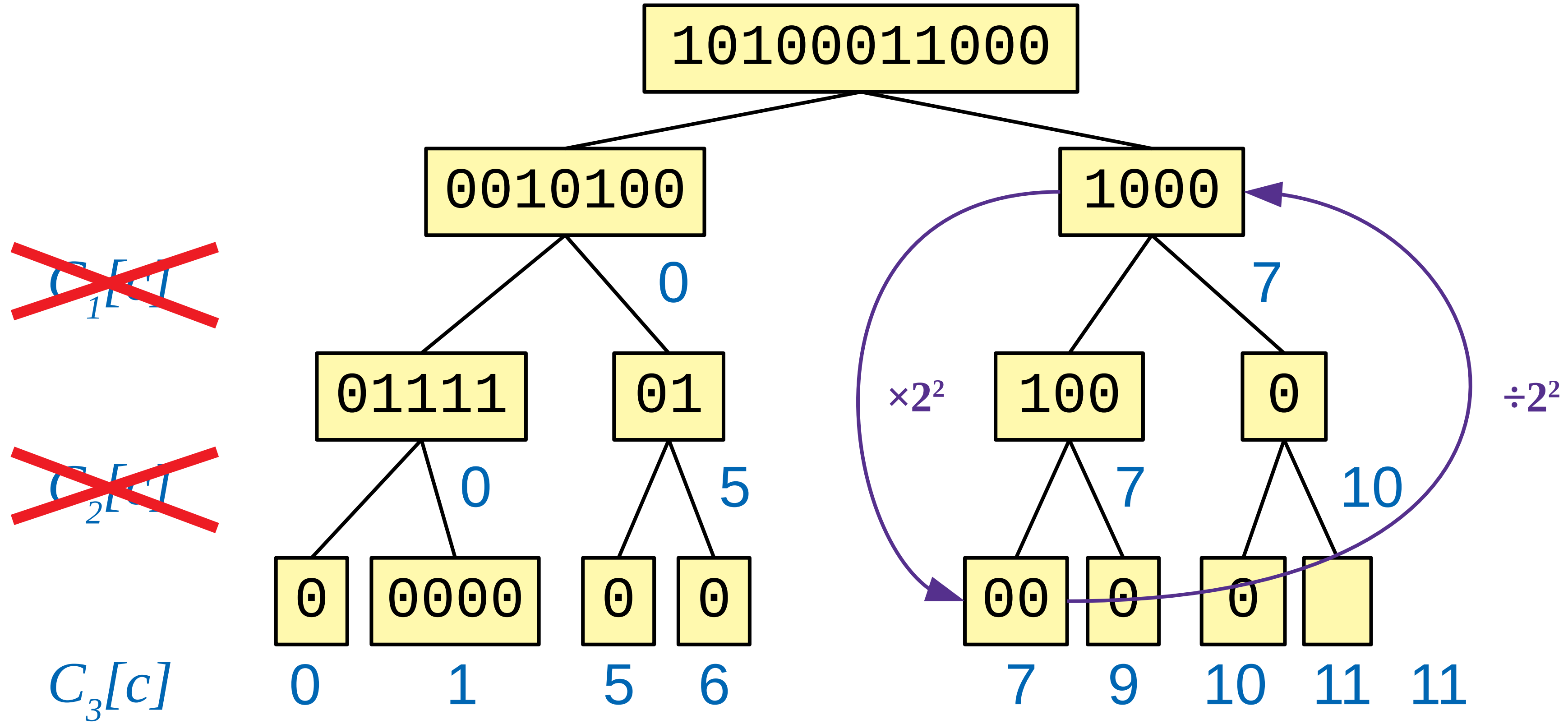

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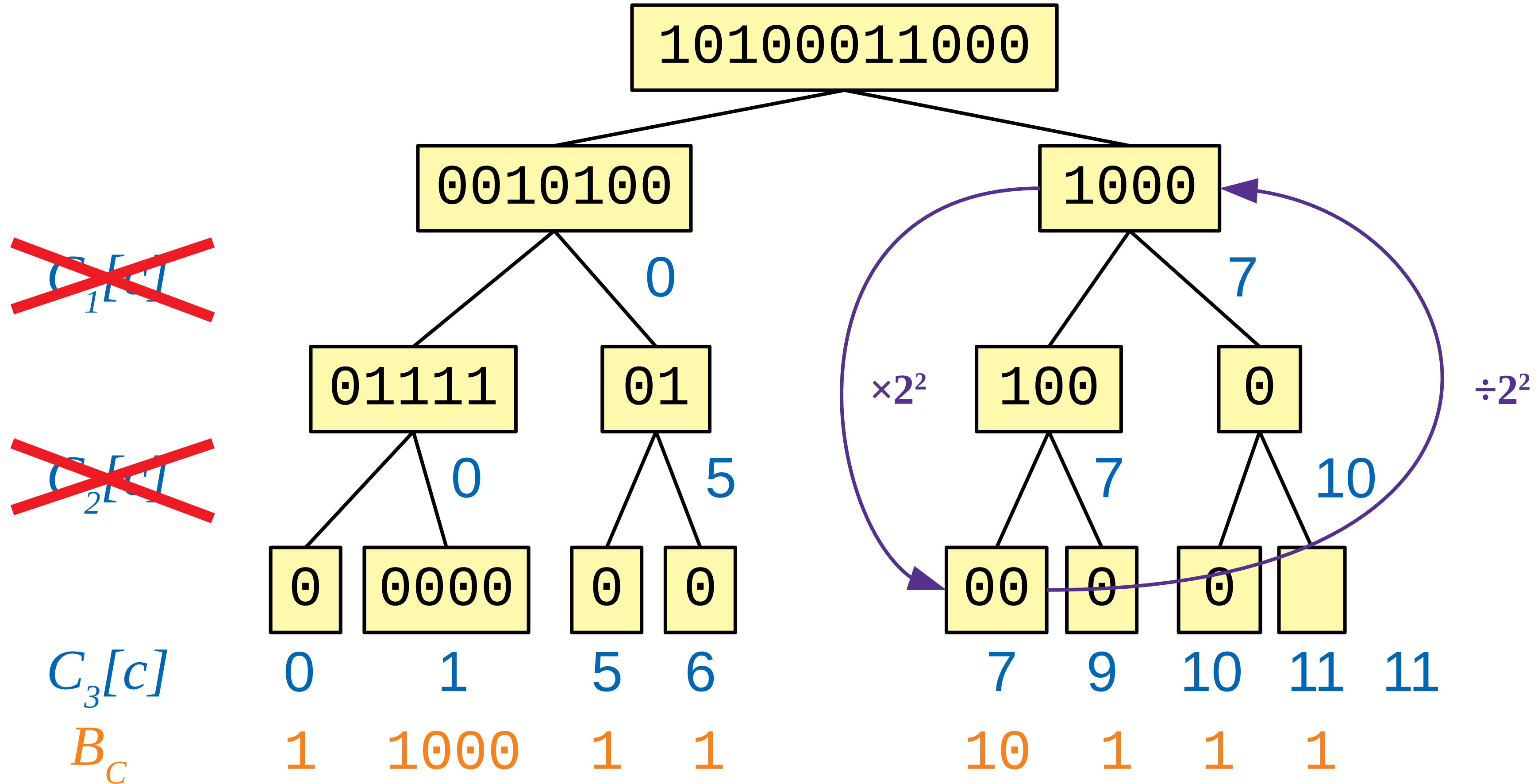

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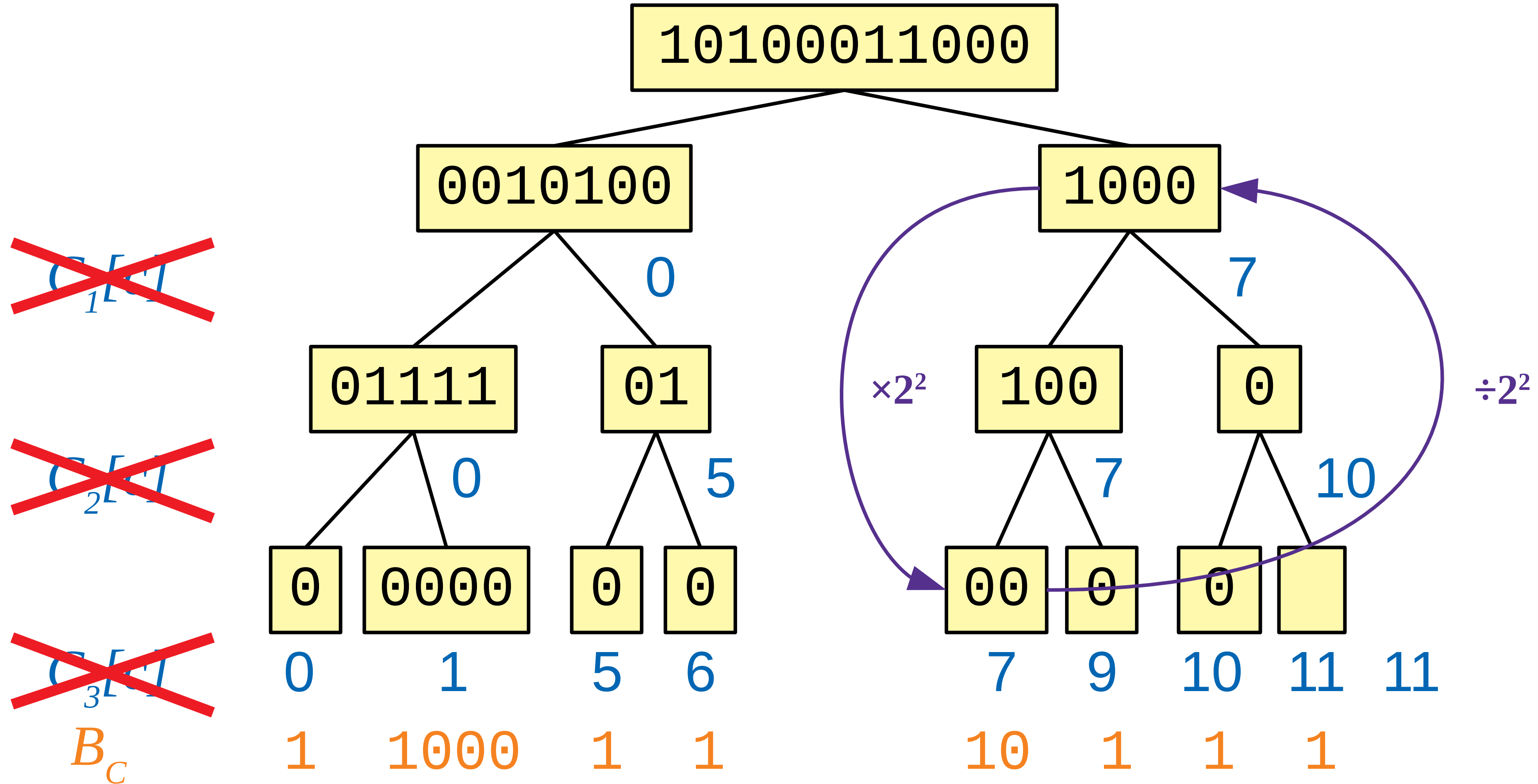

Locating in the WT
Summary

Bit vector B_c with rank/select suffices!

Locating in the WT
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Bit vector B_C with rank/select suffices!

- Construct in time $O(n+\sigma)$

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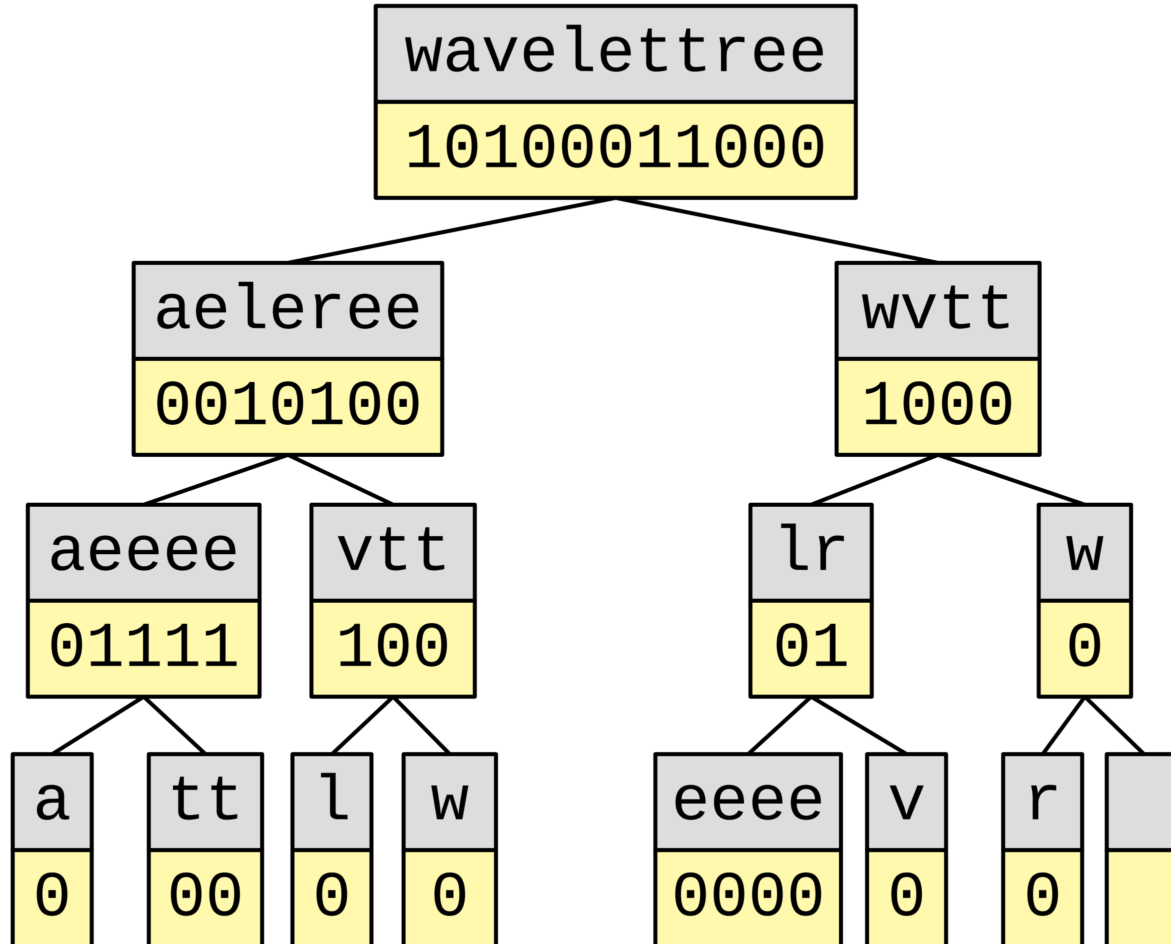
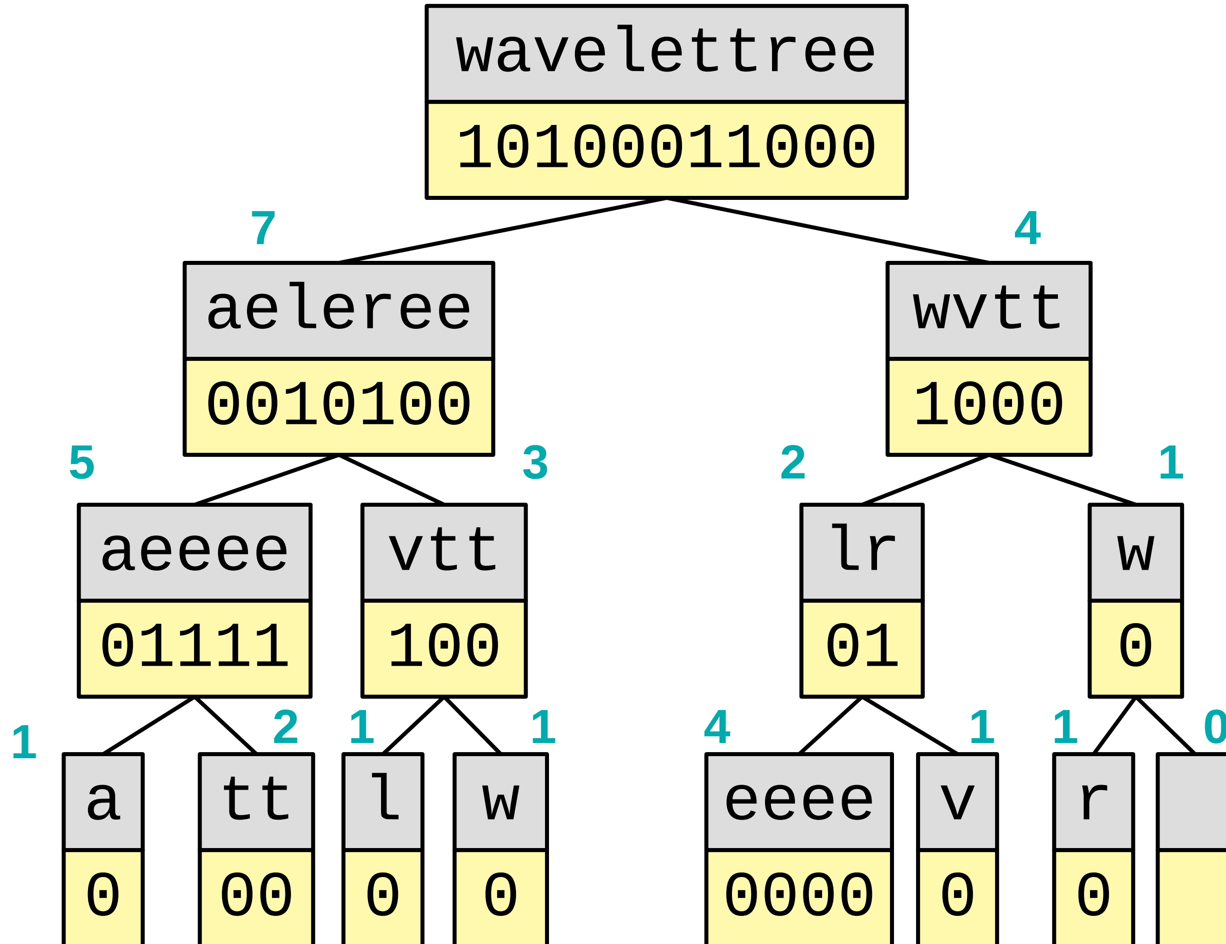
➢ Smaller than Fischer, Kurpicz & Löbel [ALENEX, 2018]

Locating in the WM

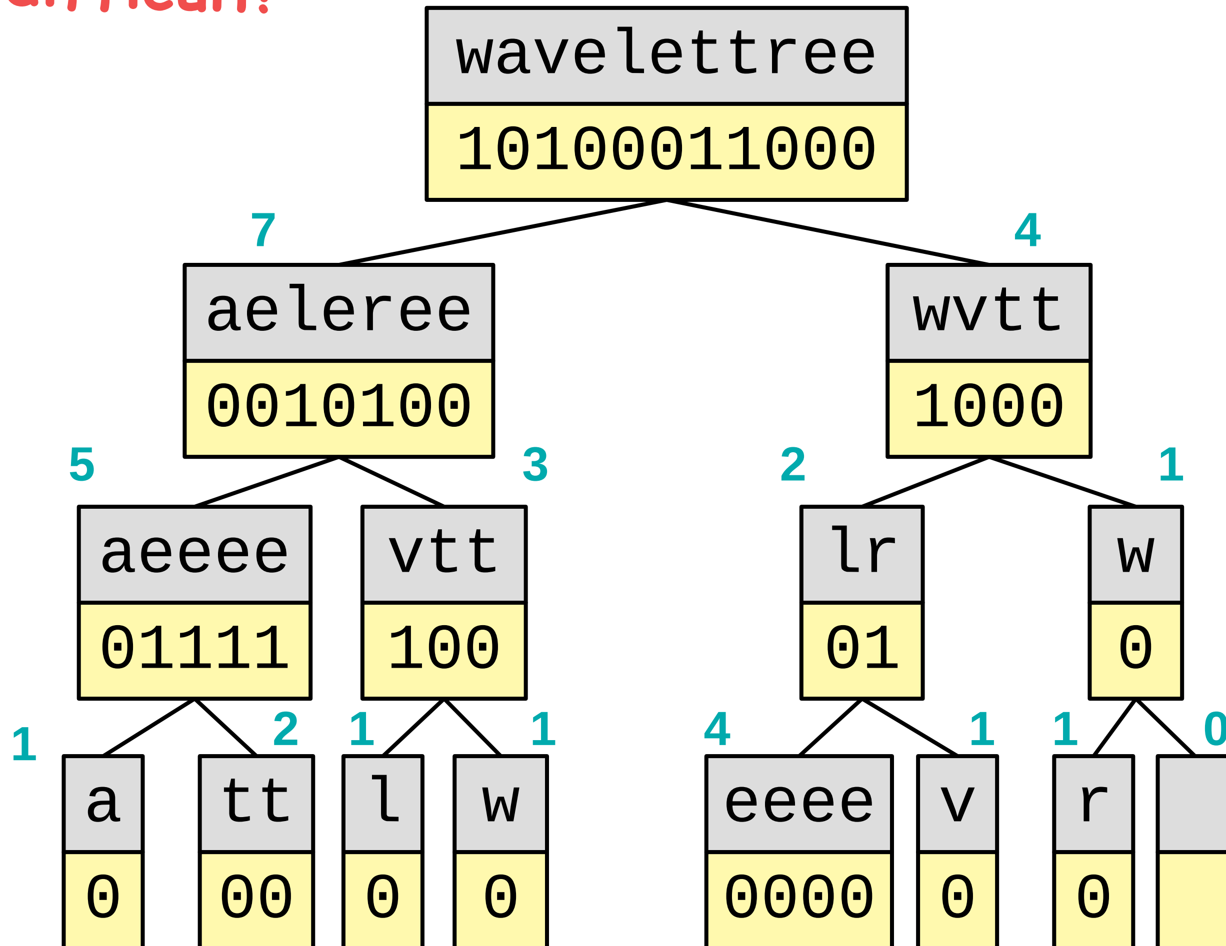


Locating in the WM



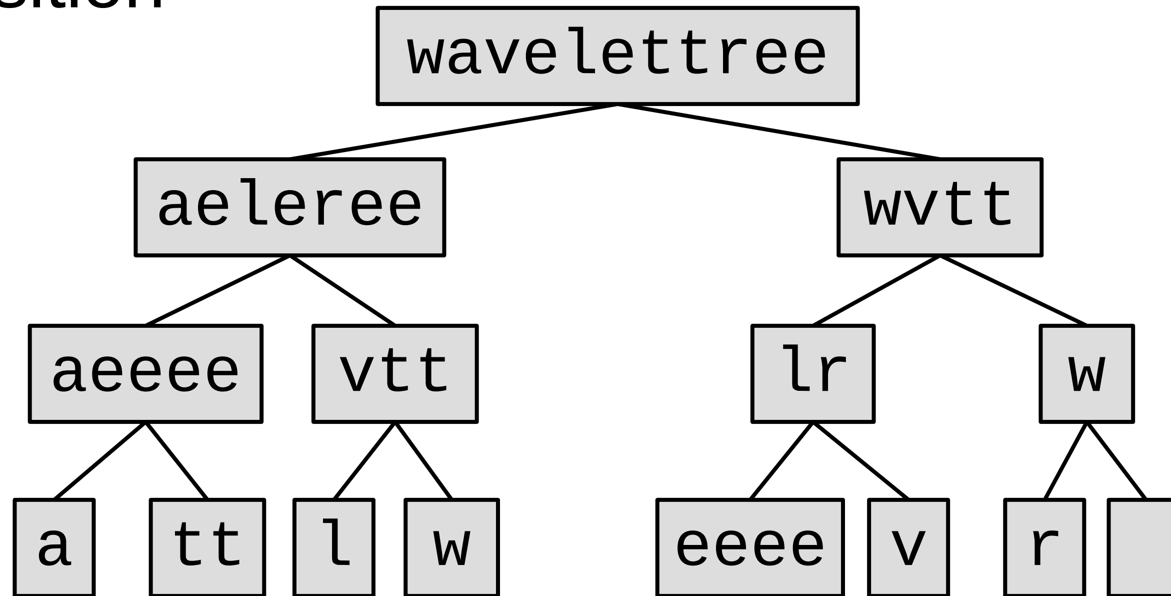
Locating in the WM

is more difficult!



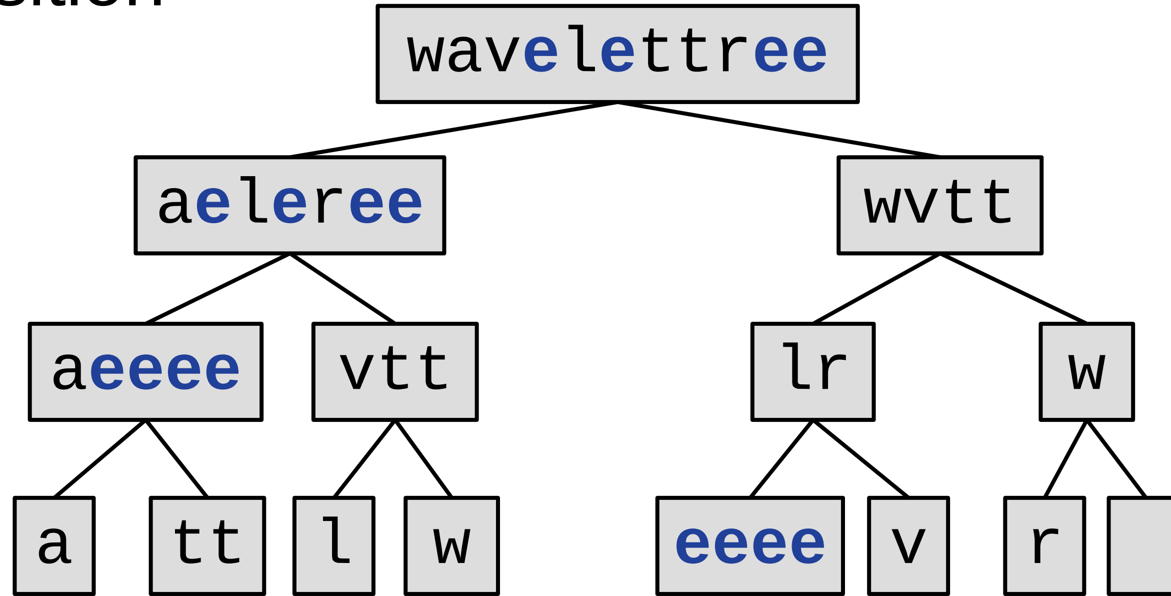
Locating in the WM

Node for position



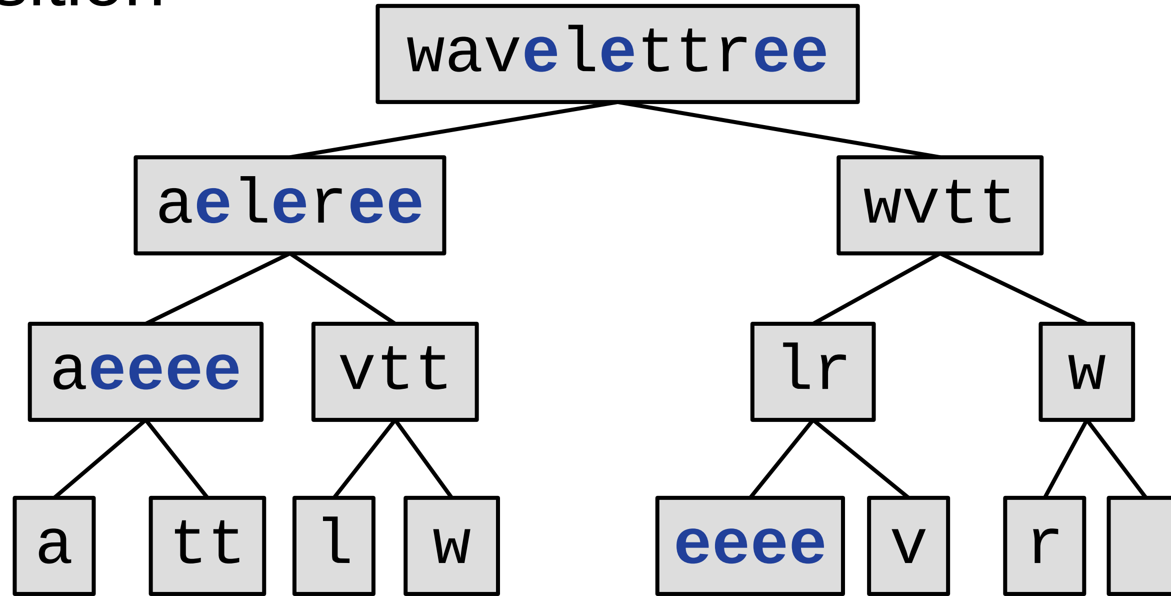
Locating in the WM

Node for position



Locating in the WM

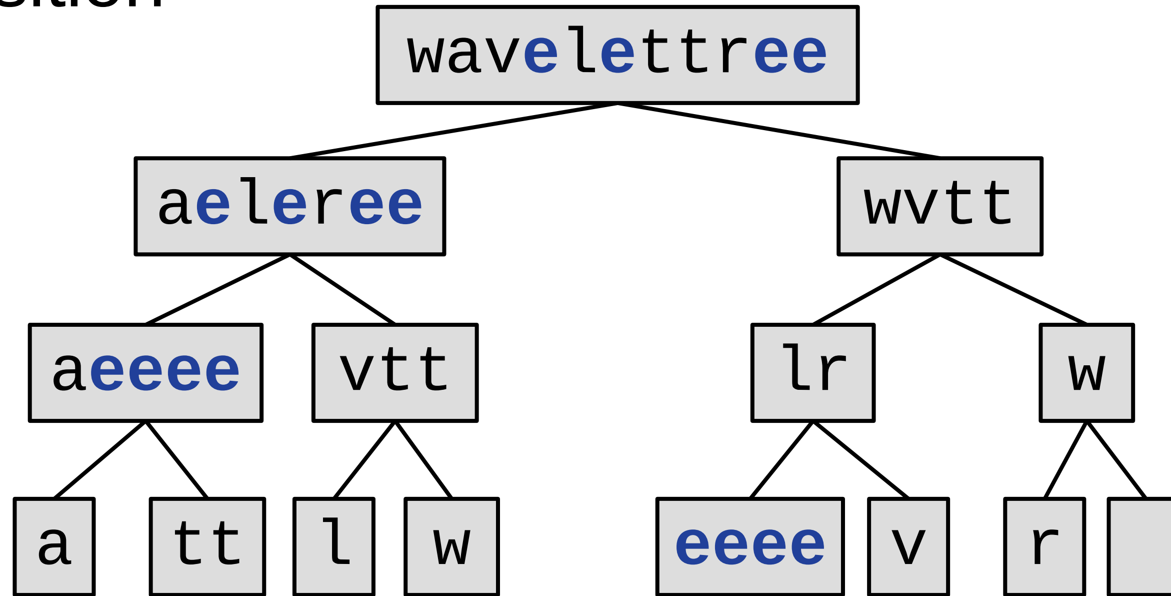
Node for position



on all levels: same symbol c = same node u

Locating in the WM

Node for position



on all levels: same symbol c = same node u

➢ if we know c when writing bit j, we only need its first occ. to find u

Locating in the WM

Node for position

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Locating in the WM

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    - in the WT, that's at $C[c]$ on the bottom level
    - we can find v in the WT using B_c

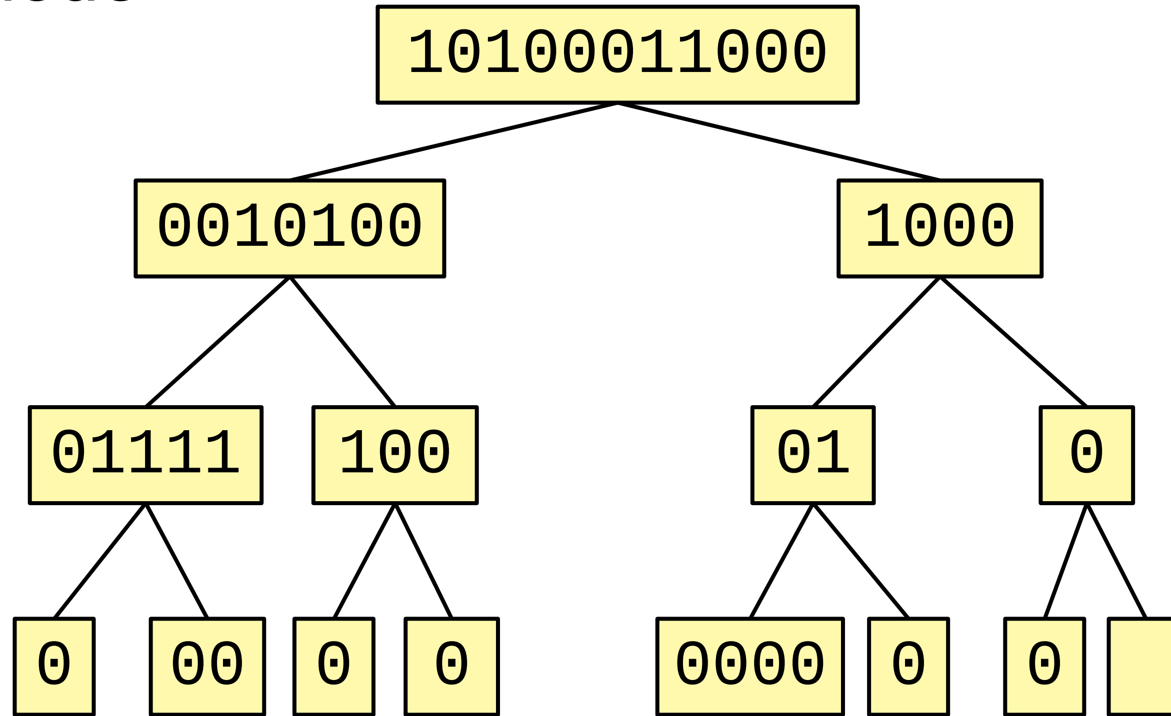
Locating in the WM

Node for position

- if we know c when writing bit j, we only need its first occ. to find u
  - in the WT, that's at $C[c]$ on the bottom level
  - we can find v in the WT using B_c
  - u is just v bit-reversed!

Locating in the WM

First bit for node



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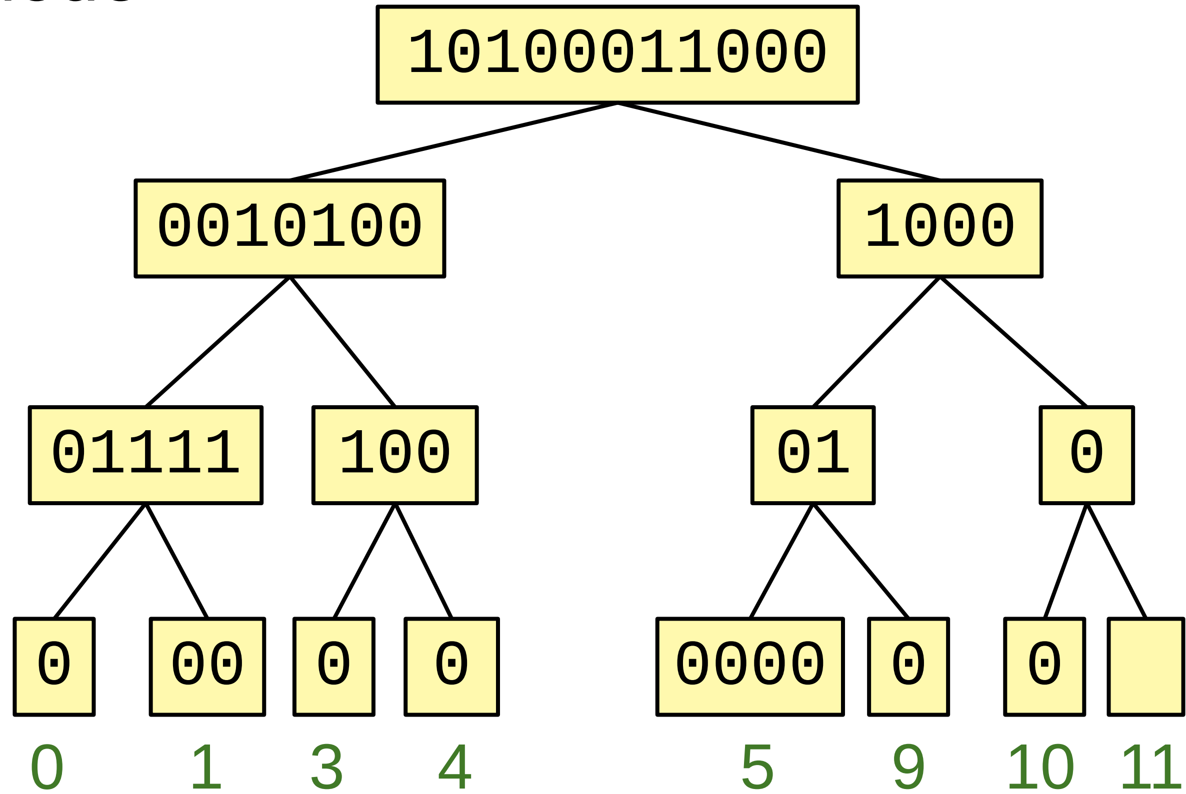

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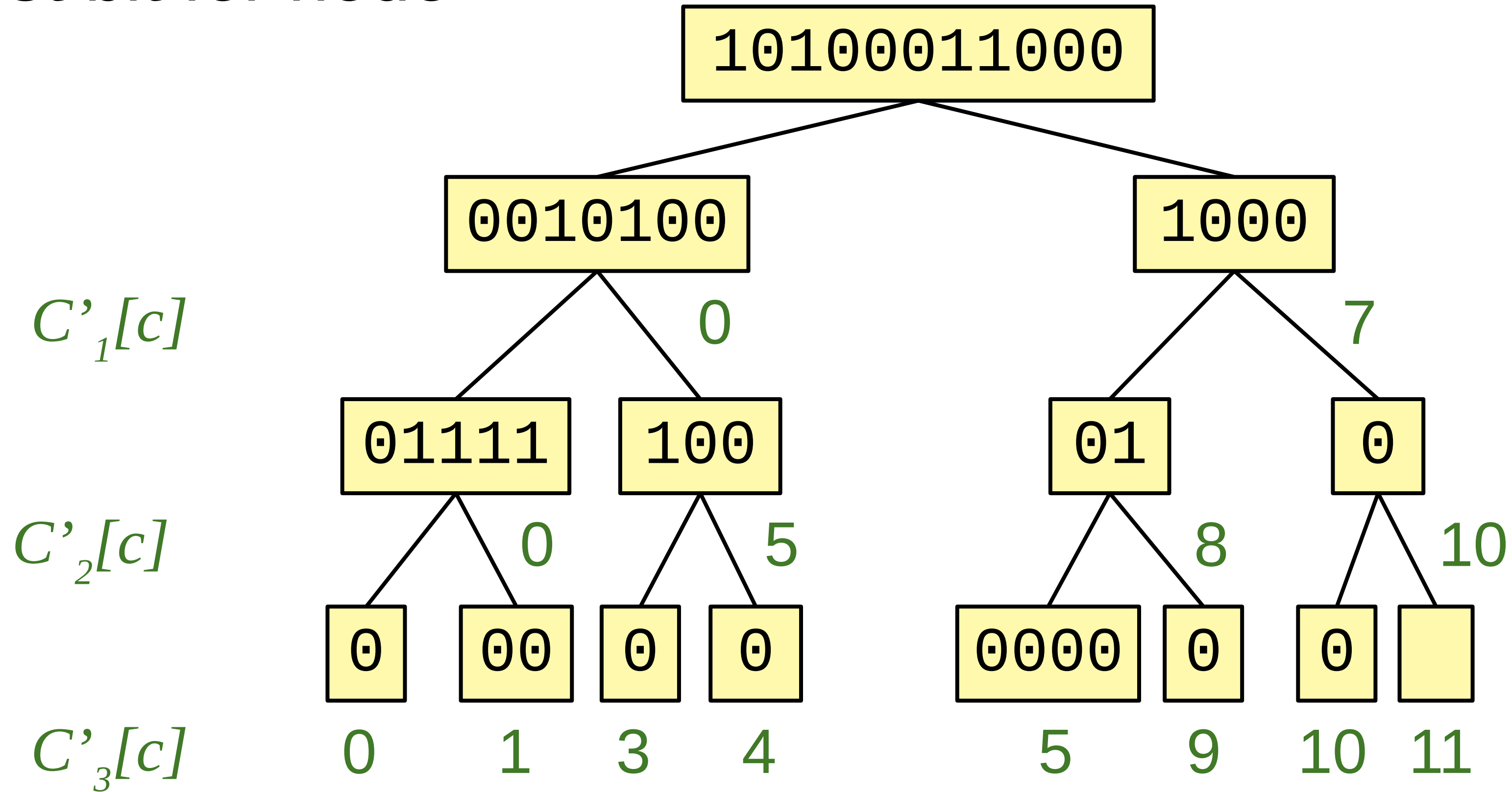

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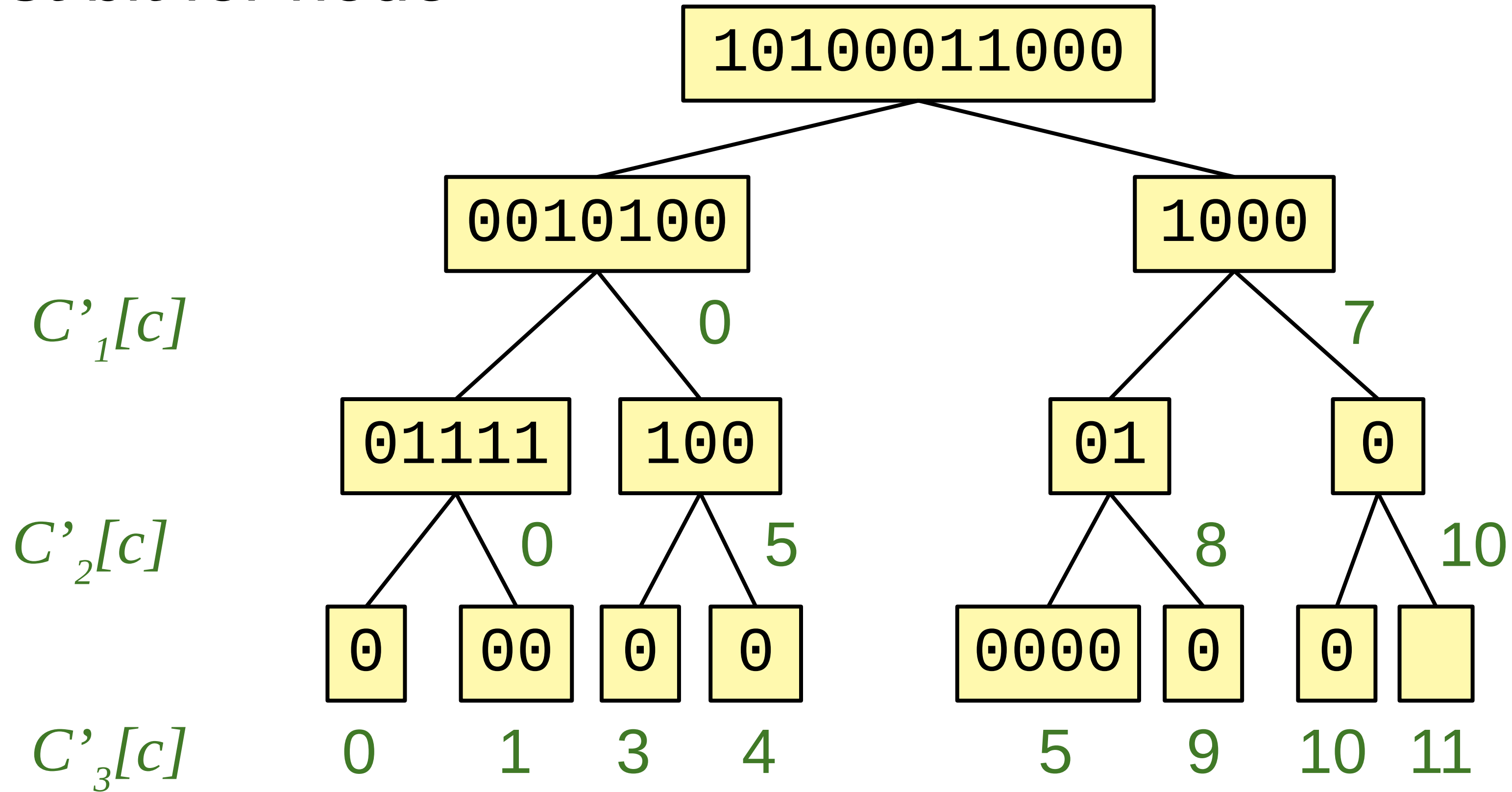

- at most $2\sigma - 1$ entries in an array ***C'***

Locating in the WM
Summary

B_C with rank/select and C' required!

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The c Oracle

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… breaks compatibility to a class of WM constructors!

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(ex.: [Kaneta, SPIRE 2018] + our translator)

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- binary search on C' to find u

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- binary search on C' to find u

  … breaks constant-time access

- bit vectors $B_{C'}$ on each level to mark node boundaries

  … makes help structure larger than WT itself

Open Questions

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- Can we improve when not using the c oracle?

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- Can we improve when not using the c oracle?
- Can we compact C'?

Conclusions

- **Translation of WT to WM** using additional $O(n+\sigma)$ time, constant overhead and $n+o(n)$ bits of space
  - *Improves previous results!*

- **Translation of WM to WT** using additional $O(n+\sigma)$ time, constant overhead and $2\sigma \lceil lg\ n \rceil + n(1+o(1))$ bits of space
  - *Requires a symbol oracle!*

- **WT seems to contain more information than the WM**