Forced repetitions over alphabet lists

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Logical foundations of String Algorithms — a formalization

Summary

- Thue's 1906 problem: construct word of arbitrary length over a ternary alphabet without repetitions.
- Grytczuk in 2010 asked if the same holds true over an *alphabet list*. Showed the answer is "yes" over alphabet lists where each alphabet has at least 4 symbols.
- Problem still open for alphabet lists with 3 symbols per alphabet. We show lots of partial results.

Thue's problem and solution

$\Sigma_3 = \{1, 2, 3\}$

morphism:

$$S = \begin{cases} 1 \mapsto 12312 \\ 2 \mapsto 131232 \\ 3 \mapsto 1323132 \end{cases} \tag{1}$$

Given a string $w \in \Sigma_3^*$, we let $S(w)$ denote w with every symbol replaced by its corresponding substitution:
$S(w) = S(w_1 w_2 \dots w_n) = S(w_1)S(w_2)\dots S(w_n)$.

Grytczuk's problem

Let:

$$L = L_1, L_2, \ldots, L_n \,,$$

be an ordered list of (finite) alphabets.

We say that w is a string over the list L if $w = w_1 w_2 \ldots w_n$ where for all i, $w_i \in L_i$.

We impose no conditions on the L_i's: they may be equal, disjoint, or have elements in common.

The only condition on w is that the i-th symbol of w must be selected from the i-th alphabet, i.e., $w_i \in L_i$.

If $|L_i| \geq 4$, then no matter what the list is, there is always a squre-free string over such a list.

This can be shown with a *probabilistic algorithm*:

> *in its i-th iteration, it selects randomly a symbol from L_i, and continues if the string created thus far is square-free, and otherwise deletes the suffix consisting of the right side of the square it just created, and restarts from the appropriate position.*

Grytczuk proves by a simple counting argument that with positive probability the length of a constructed sequence exceeds any finite bound, provided the number of symbols is at least 4.

$\Rightarrow$ This implies the existence of arbitrarily long square-free strings.

Note that this argument is *non-constructive* (or rather, it yields an exponential time procedure).

Thus it is a weaker result than the method employed in Thue's original problem.

This approach relies on Moser's algorithmic proof of *Lovász Local Lemma*: the *entropy compression argument*.

Open Problem 1:

Is there a polytime procedure for computing a square-free string from a given list of lenght n?

List of 3 symbol alphabets

$L = L_1, L_2, \ldots, L_n$, where $|L_i| = 3$.

We can prove it for restricted types of such lists:

- L has an SDR (System of Distinct Representatives)
- L has the union property
- L has a consistent mapping (testing for one is NP-hard)
- L is a partition

Offending suffix pattern

Let $\mathcal{C}(n)$, an *offending suffix*, be a pattern defined recursively:

$$\mathcal{C}(1) = \mathbf{X}_1 a_1 \mathbf{X}_1, \text{ and for } n > 1,$$
$$\mathcal{C}(n) = \mathbf{X}_n \mathcal{C}(n-1) a_n \mathbf{X}_n \mathcal{C}(n-1).$$

Zimin words ... !

We are interested in:

$$\mathcal{C}(3) = \mathbf{X}_3\mathbf{X}_2\mathbf{X}_1 a_1 \mathbf{X}_1 a_2 \mathbf{X}_2\mathbf{X}_1 a_1 \mathbf{X}_1 a_3 \mathbf{X}_3\mathbf{X}_2\mathbf{X}_1 a_1 \mathbf{X}_1 a_2 \mathbf{X}_2\mathbf{X}_1 a_1 \mathbf{X}_1,$$
$$\mathcal{C}_s(3) = a_1 a_2 a_1 a_3 a_1 a_2 a_1,$$

and observe that $\mathcal{C}_s(3)a_i$, for $i = 1, 2, 3$, all map to strings with squares.

Result

If w is a square-free string over $L = L_1, L_2, \ldots, L_{n-1}$, then $L_n = \{a, b, c\}$ forces a square iff w has suffix $\mathcal{C}(3)$.

The proof of this result is most of the PSC2016 paper.

$\Leftarrow$ direction is trivial

$\Rightarrow$ As all three squares $\mathbf{t}a\mathbf{t}a, \mathbf{u}b\mathbf{u}b, \mathbf{v}c\mathbf{v}c$ are suffixes of the string $\mathbf{w}$, it follows that $\mathbf{t}, \mathbf{u}, \mathbf{v}$ must be of different sizes, and so we can order them without loss of generality as follows:
$|\mathbf{t}a\mathbf{t}| < |\mathbf{u}b\mathbf{u}| < |\mathbf{v}c\mathbf{v}|$.

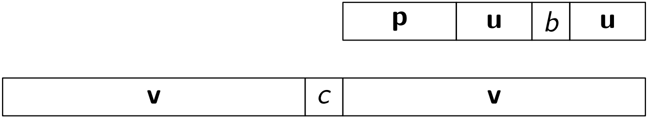
We now consider different cases of the overlap of $\mathbf{t}a\mathbf{t}, \mathbf{u}b\mathbf{u}, \mathbf{v}c\mathbf{v}$, showing in each case that the resulting string has a suffix conforming to the pattern $\mathcal{C}(3)$.

For instance:

$\mathbf{v} = \mathbf{pu}b\mathbf{u}$, where $\mathbf{p}$ is a proper non-empty prefix of $\mathbf{v}$.

Since $\mathbf{w}$ is square-free, we assume that $\mathbf{pu}b\mathbf{u}$ has no square, and therefore $\mathbf{p} \neq \mathbf{u}$ and $\mathbf{p} \neq b$.

From this, we get $\mathbf{v}c\mathbf{v} = \mathbf{pu}b\mathbf{u}c\mathbf{pu}b\mathbf{u}$. Therefore, this case is possible.



Open Problem 2:

Show that for any $L = L_1, L_2, \ldots, L_n$, where $|L_i| = 3$, there always exists a square-free w over L.

Problem: very difficult to run meaningful simulations, as the number of cases jumps up very quickly.

Hope: can we use our offending suffix characterization to do a counting argument?

Open Problem 3:

Map $L = \bigcup_i L_i \mapsto [n]$ where $n = |\bigcup_i L_i|$.

Let $M_L = (m_{ij})$ where $m_{ij} = 1 \iff j \in L_i$.

So each row if M_L will have 3 1s.

Problem: given any such matrix, can we select a single 1 from each row in such a way that there are no $2k$ consecutive rows, where the initial k rows equal the next k rows.

How to run these simulations for $n > 10$?

Crossing sequences

A clever technique in complexity to show:

- Palindromes require $\Theta(n^2)$ steps on a single tape Turing machine.
- If a language requires less then $o(\log\log n)$ memory to be decided it is in fact regular: "miniscule memory = no memory."

We use this to show that we can always avoid very large repetitions ($1/c$, $c \in \mathbb{N}$) over any list.

Perl

Truly sophisticated text manipulation libraries.

`CGI.pm` is a Perl module for CGI web applications — the workhorse of the Internet.

The text processing is done with state of the art algorithms from . . . 1970s

The advances in String Algorithms should be reflected in popular implementations.

Please visit:

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