Reducing Squares in Suffix Arrays

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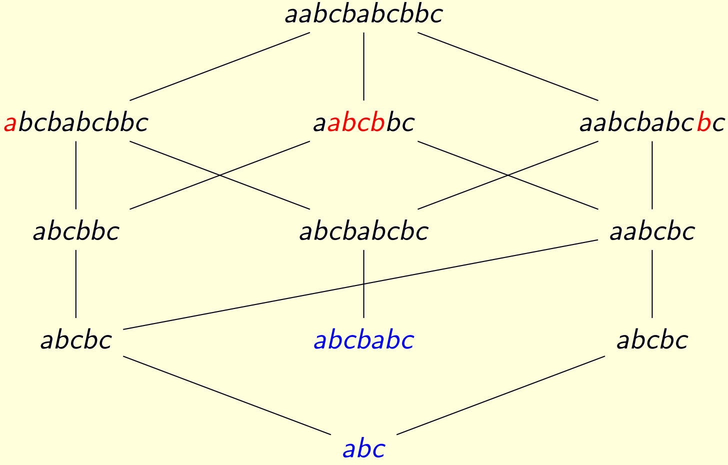
What I Really Want

The duplication history for a string *aabcbabcbbc*.
The direction of reductions is top to bottom:



Why I Want This

- For theoretical reasons (number of normal forms):

$$\mathit{duproots}(n) :=$$

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- Use duplication for phylogenetic trees
- ...

Phylogenetic Trees

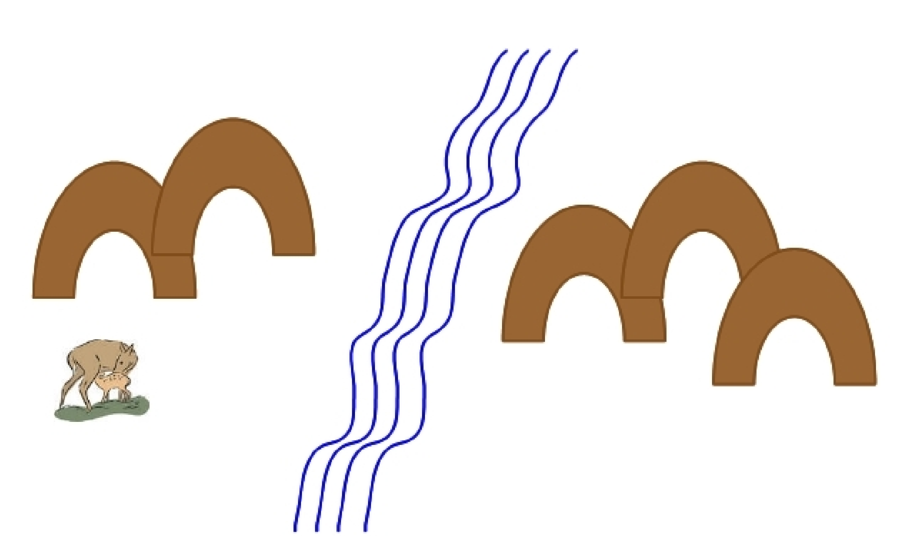
Possible duplication histories interesting for biological investigations.

In the article

> *Wapinski, I.; Pfeffer, A.; Friedman, N. and Regev, A.:*
> Natural History and Evolutionary Principles of Gene Duplication in Fungi.
> *Nature 449 (2007), pages 54–61.*

the way in which 17 populations of fungi had evolved was induced from looking at the duplication histories of their genomes.

Phylogenetic Trees



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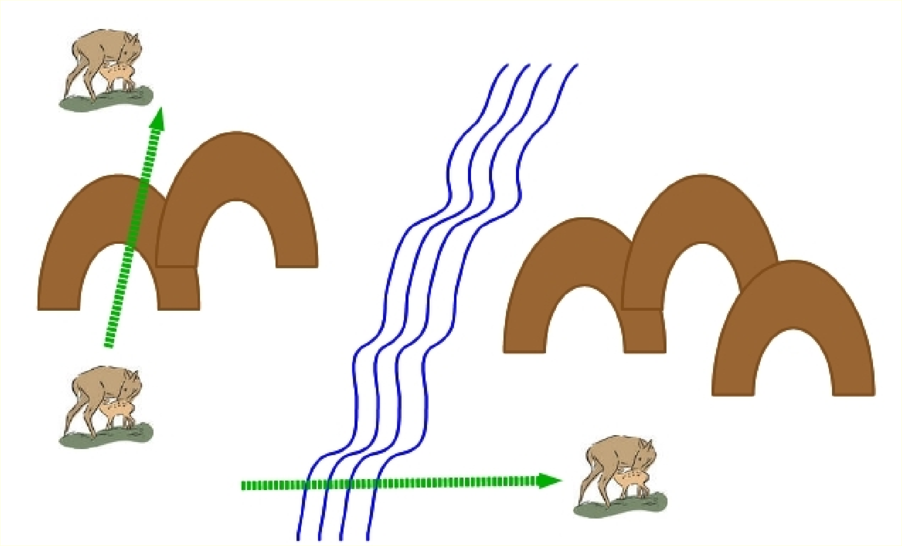

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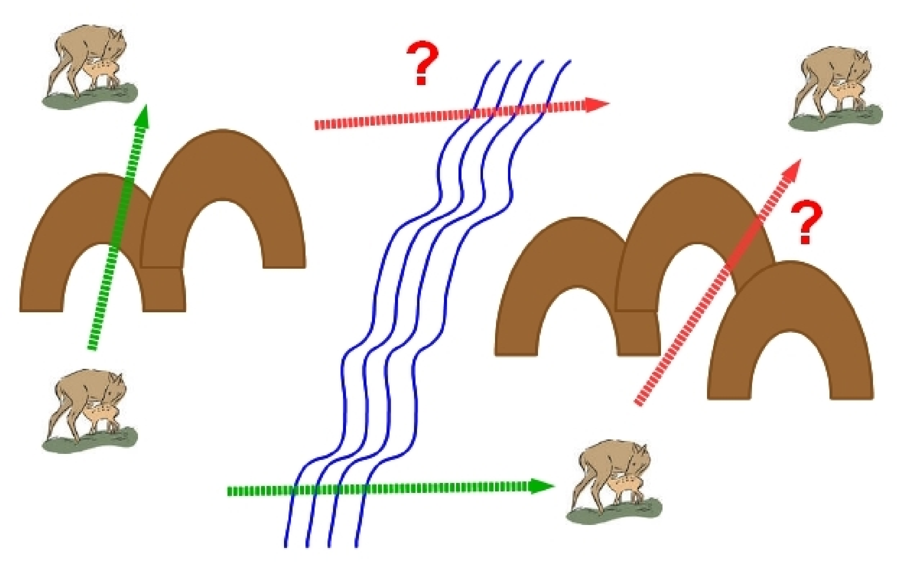

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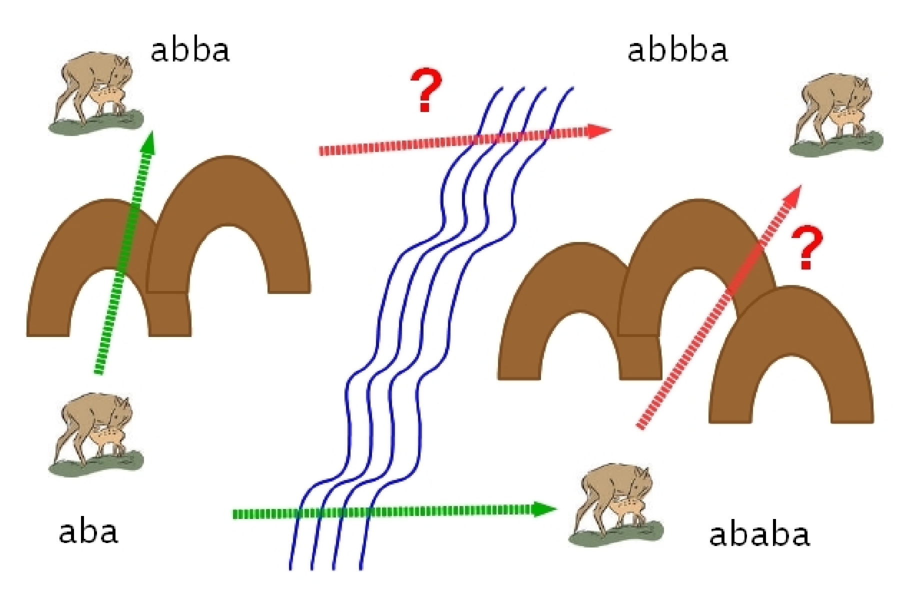

The Big Problem

Theorem (PSC 2009)

For every positive integer ℓ there are words of length ℓ over a four-letter alphabet whose number N of normal forms under eliminating squares is bounded by:

$$\frac{1}{30} 110^{\frac{\ell}{42}} \leq N \leq 2^{\ell}.$$

No efficient algorithm for all cases.

Runs, not Squares

Different square reductions in periodic factor:

abcbcbca   abcbcbca   abcbcbca

↘ ↓ ↙

abcbca

Lemma

Let w be a string with period k. Then any deletion of a factor of length k will lead to the same result.

Naive Computation of all Strings Reachable from w by Reduction of Squares

```
Input: string: w;
Data: stringlist: S (contains w);
while (S nonempty) do
    x := POP(S);
    Construct the suffix array of x;
    if (there are runs in x) then
        foreach run r do
            Reduce one square in r;
            Add new string to S;
        end
    end
    else output x;
end
```

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Modification of the suffix array by deletion of `bcb` in `abcbbcba`

SA	LCP	
7	1	`a`
0	0	`abcbbcba`
6	1	`ba`
3	1	`bbcba`
4	3	`bcba`
1	0	`bcbbcba`
5	2	`cba`
2		`cbbcba`

$\Rightarrow$

SA	LCP	
$7-3=4$	1	`a`
0	0	`abcba` (new)
$6-3=3$	1	`ba`
		—
$5-3=2$	0	`bcba`
		—
$4-3=1$		`cba`
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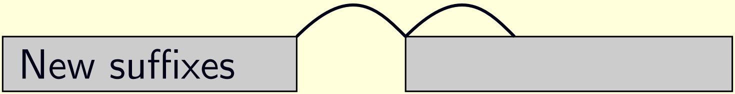
We also need: ISA

The First Small Problem



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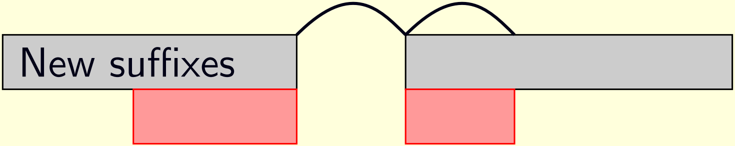

Not so new?

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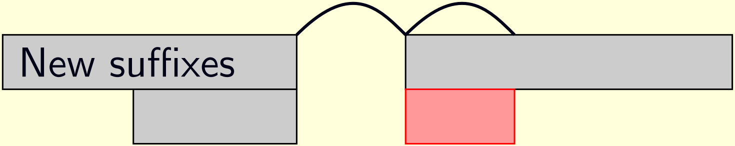


Not so new?

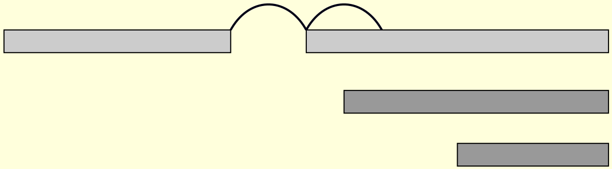
Deletions treated by:

M. Salson, T. Lecroq, M. Léonard, and L. Mouchard:
Dynamic extended suffix arrays.
J. Discrete Algorithms, 8(2) 2010, pp. 241–257.

No Change

Suffixes right of the deletion:

Order and LCP remain the same.

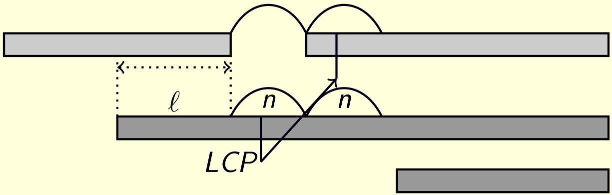


Deleting the left half of the square.

Same Order, Lower LCP

LCP is not greater than $\ell + n$:

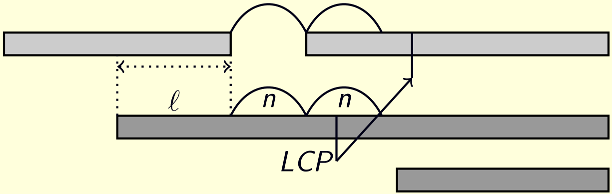
Order unchanged, because $\ell + n$ first letters remain the same.



Order Change

LCP greater than $\ell + n$:

Some letter within the prefix of length LCP might change



Computing the new suffix array

Lemma (Condition for Change in Position)

Let the LCP of two strings z and uvw be k and let $z < uvw$. Then z and $uvvw$ have the same LCP and $z < uvvw$ unless $LCP(z, uvw) \geq |uv|$; in the latter case also $LCP(z, uvvw) \geq |uv|$.

Lemma (No further changes to the left)

Let $LCP[ISA[j]] = k$ in the suffix array of a string w of length $n + 1$. Then for $i < j$ we always have $LCP[ISA[i]] \leq k + j - i$.

Computing the New Suffix Array

Input: string: w; arrays: SA, LCP;
length and pos of square: n,k;

```
for j = n + k to |w| − 1 do
|   SAnew[j]:=SA[j]-n;
end
i := k − 1;
while (LCP[i] > n + k − i AND i ≥ 0) do
|   compute SAnew of w[i ... k − 1]w[k + n ... |w| − 1];
|   compute new LCP[i];
|   /* with methods of Salson et al.             */
|   i := i − 1;
end
for j = 0 to i + m do
|   SAnew[j]:=SA[j];
end
```

Other Small Problems

- Efficient decision whether string has exponentially many ancestors
- Different examples from `abcbabcbc` with several normal forms
- Strategy for traversing the duplication history graph
- Store only changes instead of new suffix array
- Is a different method for run detection better?
- Are there strings over three letters with exponentially many normal forms?

Why I keep thinking about the LCP...