Computing Abelian periods in words

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Prague Stringology Conference 2011
31st August 2011

Outline

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Parikh vector

Definition

$\mathcal{P}(w)$: Parikh vector of a word w, that enumerates the cardinality of each letter of the alphabet in w

Definition

$\mathcal{P}_w(i,k)$: Parikh vector of the factor of w starting at position i and of length k

Example with $w =$ abaababa

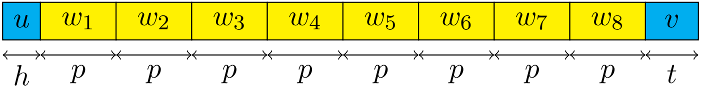
$\mathcal{P}(\textbf{abaababa}) = (5, 3)$
$\mathcal{P}_w(2,3) = (2,1)$ ($w =$ **abaababa**)

Abelian periods

Definition

A word w has an Abelian period (h, p) if

- $w = uw_1 \cdots w_k v$
- $|u| = h$, $|w_i| = p$ for all i
- $\mathcal{P}(u) \subset \mathcal{P}(w_1) = \cdots = \mathcal{P}(w_k) \supset \mathcal{P}(v)$



u (or h) is called the head and v (or t) is called the tail

weak repetitions or *Abelian powers* when $u, v = \varepsilon$ and $k \geq 2$

Abelian periods

Example

$(2, 3)$ is an Abelian period of **ab** $\cdot$ **aab** $\cdot$ **aba** with $\mathcal{P} = (2, 1)$

$(1, 2)$, $(0, 3)$, $(2, 3)$, $(1, 4)$, $(2, 4)$, $(3, 4)$, $(0, 5)$, $(1, 5)$, $(2, 5)$, $(3, 5)$, $(0, 6)$, $(1, 6)$, $(2, 6)$, $(0, 7)$, $(1, 7)$ and $(0, 8)$ are all the Abelian periods of **abaababa**

Periods

Classical: well studied

Abelian: properties but only one algorithm

Maximal number of Abelian periods

Lemma

The maximal number of Abelian periods of a word of length n is $O(n^2)$.

Proof

a^n has

- n periods with head length 0
- $n-2$ periods with head length 1
- $\vdots$
- 0 or 1 periods with head length $\lfloor n/2 \rfloor$ depending on the parity of n

Outline

Brute-force algorithm in $O(n^3 \times \sigma)$

```
BruteForce(w, n)
for all possible head h do
    for all possible p > h such that p + h ⩽ n do
        if (h, p) is an Abelian period of w then
            Output(h, p)
```

$select$ array

The $select$ array is defined as follows: $select[a, i]$ is equal to the i-th position of letter a in w. For example: $w =$agcgatacagatctagcgact

	1	2	3	4	5	6	7
'a'	1						
'c'							
'g'							
't'							

select array

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'c'	3	8	13	17	20		
'g'	2	4	10	16	18		
't'	6	12	14	21			

select function: $O(n^2) \to O(n+\sigma)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
w	a	g	c	g	a	t	a	c	a	g	a	t	c	t	a	g	c	g	a	c	t

	'a'	'c'	'g'	't'
ind	1	2	3	4

	1	2	3	4	5
C	1	8	13	18	22

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
S	1	5	7	9	11	15	19	3	8	13	17	20	2	4	10	16	18	6	12	14	21

$$\forall a \in \Sigma, i \leqslant |w|_a, select_a(w,i) = S[C[ind[a]] + i - 1]$$

The second 'c' in w appears at index

$select_{\text{c}}(w,2) = S[C[ind[\text{c}]] + 2 - 1] = 8$

$select$ function: $O(n^2) \rightarrow O(n+\sigma)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
w	a	g	c	g	a	t	a	c	a	g	a	t	c	t	a	g	c	g	a	c	t

	'a'	'c'	'g'	't'
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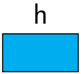
Improvements

Given h



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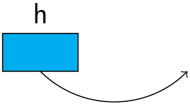
Given h, what is the minimal p such that (h, p) can be an Abelian period of w?



Using the $select$ function, one can find the minimal s such that $\mathcal{P}_w(1, h) \subset \mathcal{P}_w(h + 1, s)$.

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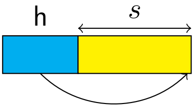

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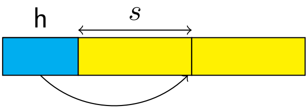

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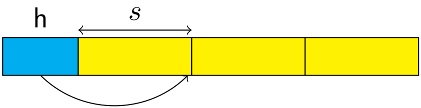

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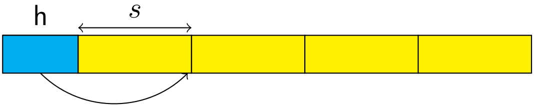

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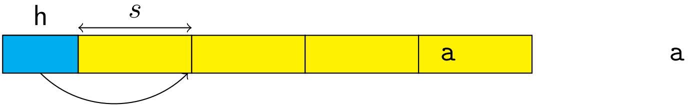

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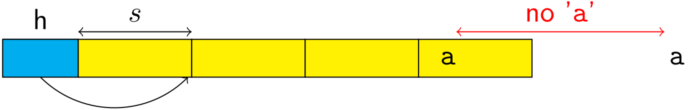

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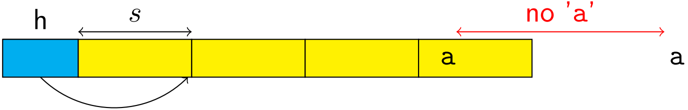

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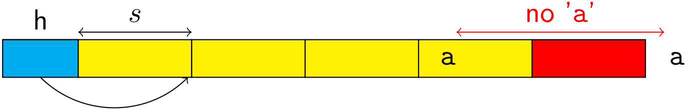

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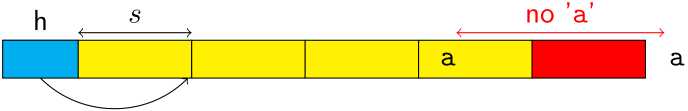

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S	1	5	7	9	11	15	19	3	8	13	17	20	2	4
G														

$$G[h] = max\{S[i+1] - S[i] | h < i < |w|\}$$

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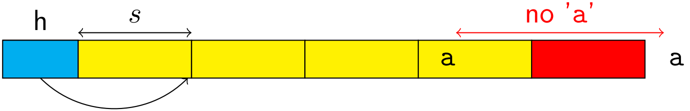

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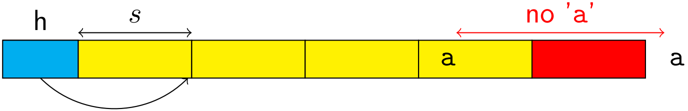

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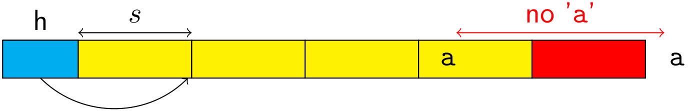

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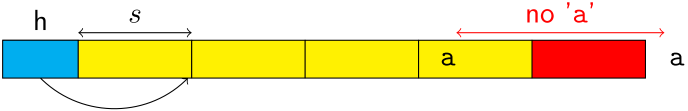

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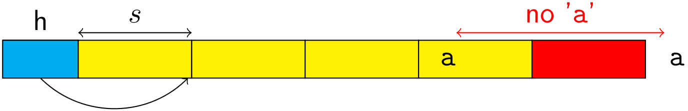

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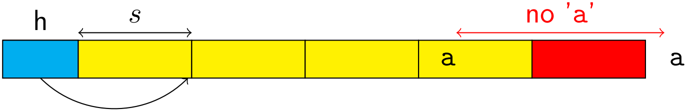

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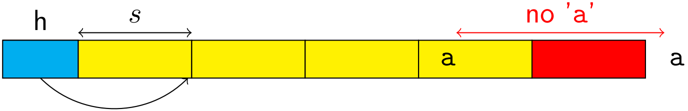

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Find the minimal p

For a given h: The minimal p value such that (h, p) can be an Abelian period of w is the maximum between s, $G[h+1]/2$ and $h+1$.

Find the minimal p

For a given h: The minimal p value such that (h, p) can be an Abelian period of w is the maximum between s, $G[h+1]/2$ and $h+1$.

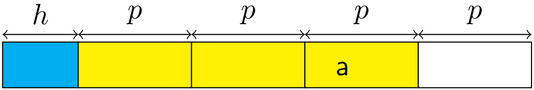
Shift function

Once (h, p) is fixed, a Shift function verifies whether it corresponds to an Abelian period or not.

Shift function

The Shift function is based on the *select* function.

'a'	2	$\cdots$	10	15	16	20	$\cdots$



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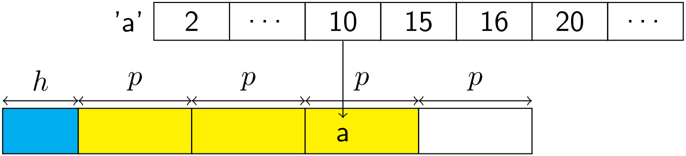

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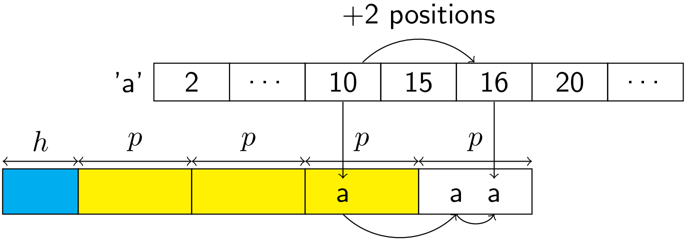
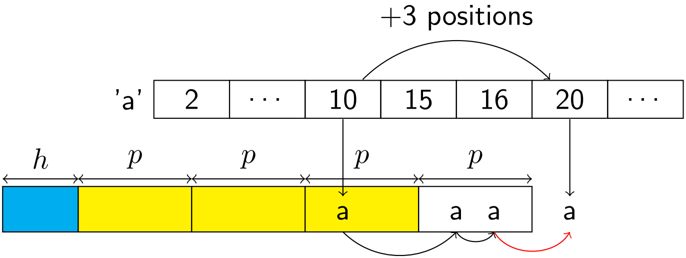


if the number of a's in the Parikh vector is 2, it's ok!

Shift function

The Shift function is based on the *select* function.



if the number of a's in the Parikh vector is 3, it's not ok!

Complexity

Time: $O(n^2 \times \sigma)$

Space: $O(n^2 + \sigma)$

Outline

On-line algorithms

Proposition

If (h, p), $h + p \leqslant |w|$, is not an Abelian period of w then (h, p) is not an Abelian period of wa for any symbol $a \in \Sigma$.

$\Longrightarrow$ skip some periods with an on-line algorithm.

Dynamic programming

When processing position i:
$T[h, p] = j$ iff $w[1\,.\,.\,j]$ is the longest prefix of $w[1\,.\,.\,i]$ having Abelian period (h, p).

Dynamic programming

When processing position i:
$T[h,p] = j$ iff $w[1..j]$ is the longest prefix of $w[1..i]$ having Abelian period (h,p).

$w =$ **abaababa**

1

a

$h \backslash p$	1	2	3	4	5	6	7	8
0	1							
1								
2								
3								

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1	2
a	**b**

$h \backslash p$	1	2	3	4	5	6	7	8
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$w =$ **abaababa**

1	2	3
a	**b**	**a**

$h \backslash p$	1	2	3	4	5	6	7	8
0	1	3	3					
1		3						
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$w =$ **abaababa**

1	2	3	4
a	**b**	**a**	**a**

$h \backslash p$	1	2	3	4	5	6	7	8
0	1	3	4	4				
1		4	4					
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Dynamic programming

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$w =$ **abaababa**

1	2	3	4	5
a	**b**	**a**	**a**	**b**

$h \backslash p$	1	2	3	4	5	6	7	8
0	1	3	5	5	5			
1		5	5	5				
2			5					
3								

Dynamic programming

When processing position i:
$T[h,p] = j$ iff $w[1..j]$ is the longest prefix of $w[1..i]$ having Abelian period (h,p).

$w =$ **abaababa**

1	2	3	4	5	6
a	**b**	**a**	**a**	**b**	**a**

$h \backslash p$	1	2	3	4	5	6	7	8
0	1	3	6	6	6	6		
1		6	6	6	6			
2			6	6				
3								

Dynamic programming

When processing position i:
$T[h,p] = j$ iff $w[1..j]$ is the longest prefix of $w[1..i]$ having Abelian period (h,p).

$w =$ **abaababa**

1	2	3	4	5	6	7	8
a	**b**	**a**	**a**	**b**	**a**	**b**	

$h \backslash p$	1	2	3	4	5	6	7	8
0	1	3	7	6	7	7	7	
1		7	6	7	7	7		
2			7	7	7			
3				7				

Dynamic programming

When processing position i:
$T[h,p] = j$ iff $w[1..j]$ is the longest prefix of $w[1..i]$ having Abelian period (h,p).

$w =$ **abaababa**

1	2	3	4	5	6	7	8
a	**b**	**a**	**a**	**b**	**a**	**b**	**a**

$h \backslash p$	1	2	3	4	5	6	7	8
0	1	3	8	6	8	8	8	8
1		8	6	8	8	8	8	
2			8	8	8	8		
3				8	8			

Lists

When processing position i we only store couples (h, p) which are Abelian periods of $w[1..i]$.

Complexity

Both algorithms

Time: $O(n^3 \times \sigma)$

Space: $O(n^2)$

where

- n is the length of w
- σ is the size of the alphabet

Heaps

Proposition

w has s Abelian periods $(h_1, p_1) < (h_2, p_2) < \cdots < (h_s, p_s)$ such that $(|w| - h_i) \mod p_i = t > 0$ for $1 \leq i \leq s$.
If (h_1, p_1) is an Abelian period of wa for any symbol $a \in \Sigma$ then $(h_2, p_2), \ldots, (h_s, p_s)$ are also Abelian periods of wa.

Heaps

One heap will be created for each position of w.

Heaps

Proposition

w has s Abelian periods $(h_1, p_1) < (h_2, p_2) < \cdots < (h_s, p_s)$ such that $(|w| - h_i) \mod p_i = t > 0$ for $1 \le i \le s$.
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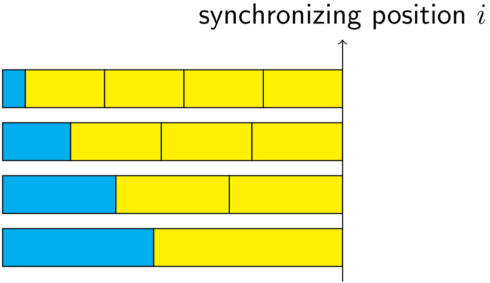
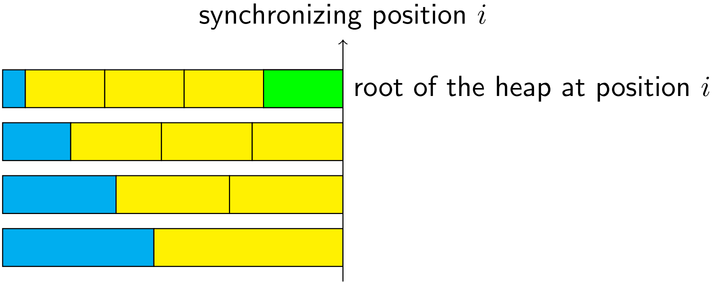
Heaps

One heap will be created for each position of w.

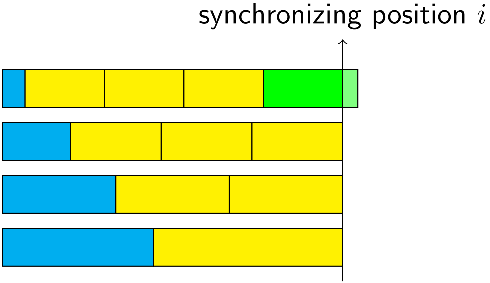
Synchronizing positions and heaps



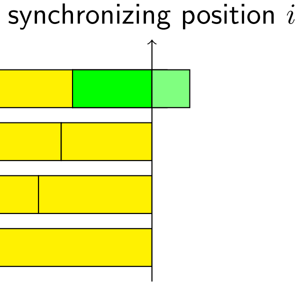
Synchronizing positions and heaps



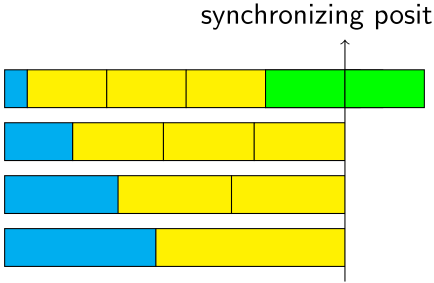
Synchronizing positions and heaps



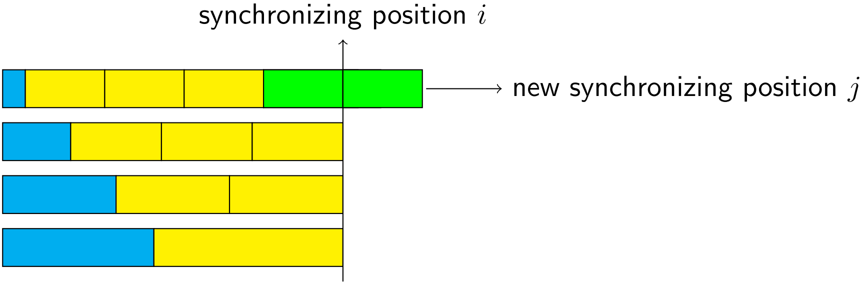
Synchronizing positions and heaps



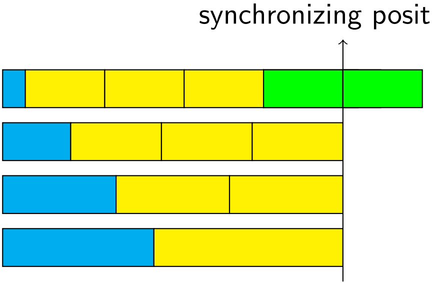
Synchronizing positions and heaps



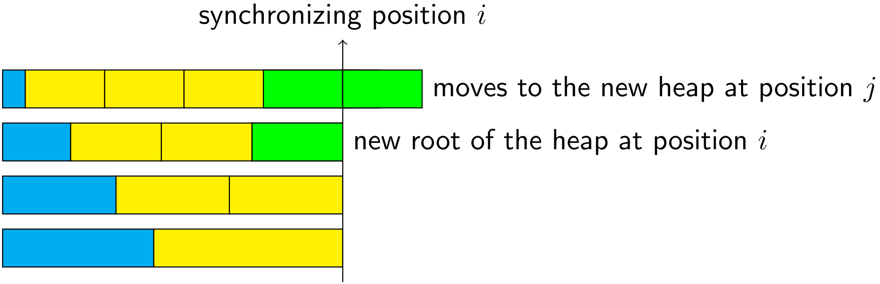
Synchronizing positions and heaps



Synchronizing positions and heaps



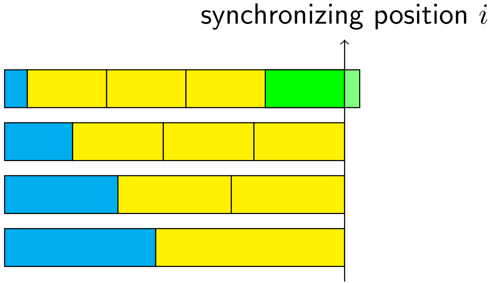
Synchronizing positions and heaps



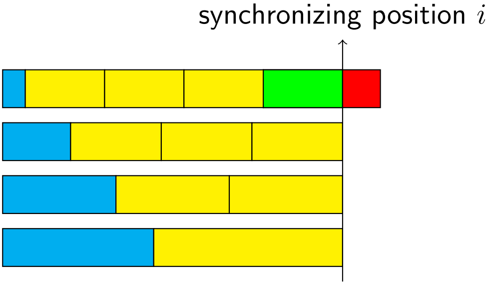
New heaps

Each new heap created at each position j is initialized with all the "trivial" periods (h, p) such that $h + p = j$ and $\mathcal{P}(w[1..h]) \subset \mathcal{P}(w[h+1..j])$.

Synchronizing positions and heaps



Synchronizing positions and heaps



Synchronizing positions and heaps

Example

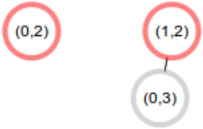
$$w = \mathtt{a}$$

Example

$w = \texttt{ab}$

Example

$$w = \texttt{aba}$$



Example

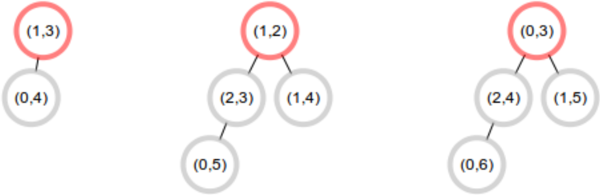
$w = \texttt{abaa}$

Example

$$w = \texttt{abaab}$$

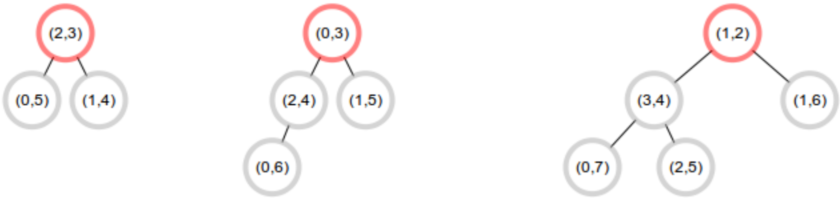
Example

$w = \texttt{abaaba}$



Example

$$w = \texttt{abaabab}$$



Example

$w =$ `abaababa`

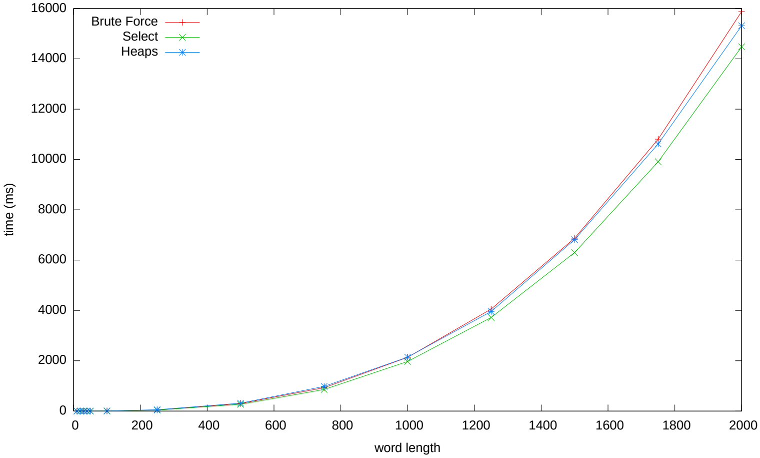
Complexity

Time: $O(n^2 \times (n \log n) \times \sigma)$
Space: $O(n^2)$

Outline

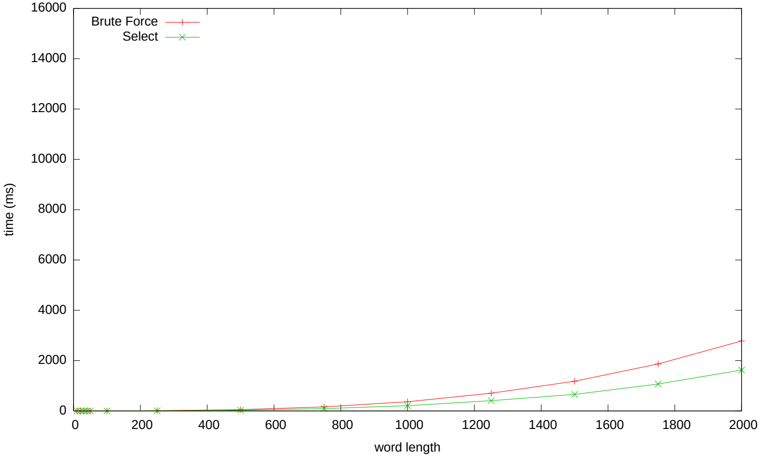
Results

Running times on alphabet size 2 and on random words of lengths 10 up to 2000.



Results

Running times on alphabet size 2 and on random words of lengths 10 up to 2000. Only computing (h, p) such that $h + 2p \leqslant |w|$.



Outline

Conclusion

Conclusion

Different algorithms to compute all the Abelian periods of a word

Perspectives

Set of cutting positions

$$C(h,p) = \{h + kp \mid k \geq 0 \text{ and } h + kp \leq |w|\}$$

Non-deducible Abelian period

An Abelian period (h, p) is non-deducible for the word w if for any other Abelian period (h_0, p_0) of w one has $C(h, p) \nsubseteq C(h_0, p_0)$.
Otherwise (h, p) is deducible.

Perspectives

Perspectives

- improve complexity
- compute the non-deducible Abelian periods
- Abelian periods in Fibonacci words
- Abelian periods in Sturmian words
- compute the Abelian border array
- applications such as looking for repeated regions with same nucleotides composition

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