Finding all covers of an indeterminate string in O(n) time on average

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Outline

- Introduction
 - Definitions
 - The Problems That We Studied
 - Previous Works
- Our Contributions
 - Main Results
 - Our Algorithm





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Basic Definitions

For a strings x = uwv:

- |x| is the **length** of x
- ullet is the **empty** string
- x[i] is the i-th symbol of x
- w is a substring of x and x is a superstring of w
- u(v) is a **prefix (suffix)** of x
- x[i...j] denotes the substring of x starting at position i and ending at j



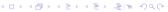


Basic Definitions

For strings $x = x[1 \dots n]$ and $y = y[1 \dots m]$:

- *xy* denotes the **concatenation** of strings *x* and *y*.
- x^k denotes the concatenation of k copies of x.
- If x[n-i+1...n] = y[1...i] for some $i \ge 1$, the string x[1...n]y[i+1...m] is a **superposition** of x and y. We also say that x overlaps y.





Border and Border Array

Border and Border Array:

- A **border** *u* of *x* is a prefix of *x* that is also a suffix of *x*.
- That is u = x[1 ... b] = x[n b + 1 ... n] for some $b \in \{0 ... n 1\}$.
- The border array of x is an array β such that for all $i \in \{1 \dots n\}$, $\beta[i] = \text{length of the longest border of } x[1 \dots i].$





Cover and Cover Array

Cover and Cover Array:

- A substring w of x is called a cover of x, if x can be constructed by concatenating or overlapping copies of w. We also say that w covers x.
- For example, if x = ababaaba, then aba and x are covers of x.
- The cover array γ, is a data structure used to store the length of the longest proper cover of every prefix of x;
- That is for all $i \in 1 \dots n$, $\gamma[i] = \text{length of the longest proper cover of } x[1 \dots i] \text{ or } 0$.





Indeterminate Strings

Indeterminate Strings:

- An indeterminate string is a sequence T = T[1]T[2]...T[n], where $T[i] \subseteq \Sigma$ for each i, and Σ is a given alphabet of fixed size.
- If at any position in an indeterminate string, |T[i]| = 1, we call this a **solid symbol**. However, when $|T[i]| \ge 1$, we call this a **non-solid symbol**.
- In an indeterminate string a non-solid position can contain up to $|\Sigma|$ symbols.





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The problems that we studied here are

- Finding all the covers of an indeterminate string
- Finding the cover array of an indeterminate string





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Regularities of conservative indeterminate strings

- In [1], the authors investigated the regularities of conservative indeterminate strings.
- In a conservative indeterminate string the number indeterminate positions is bounded by a constant.
- The authors presented algorithms for finding
 - The smallest conservative cover (number of indeterminate position in the cover is bounded by a given constant)
 - λ -conservative covers (conservative covers having a fixed length λ)
 - λ -conservative seeds.





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Regularities on indeterminate strings (without any restriction)

- Antoniou et al. presented an O(n log n) algorithm to find the smallest cover of an indeterminate string in [2].
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- We devise an algorithm for computing all the covers of an indeterminate string x of length n in $O(n^2)$ time in the worst case.
- We also show that our algorithm works in O(n) time on the average.
- We extend our algorithm to compute the cover array of x in $O(n^2)$ time and O(n) space complexity in the worst case.
- Notably, our algorithm, unlike the algorithm of [1], does not enforce the restriction that the cover or the input string x must be conservative.





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Definition of the The First Problem

We start with a formal definition of the first problem we handle in this paper.

Problem

Computing All Covers of an Indeterminate String over a fixed alphabet.

Input: We are given an indeterminate string x, of length n on a fixed alphabet Σ .

Output: We need to compute all the covers of x.





Our Algorithm Depends on the Following Facts

Fact

Every cover of string x is also a border of x.

Fact

If u and c are covers of x and |u| < |c| then u must be a cover of c.





Our Algorithm Depends on the Following Lemma

Lemma

The expected number of borders of an indeterminate string is bounded by a constant.





The Algorithm

- In the first step, the deterministic border array of *x* is computed.
- In the second step, we check each border whether it is a cover of x or not.





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- Here, we utilize the algorithm provided by Holub and Smyth [3] for computing the deterministic border of an indeterminate string.
- The output of the algorithm is a two dimensional list β .
- Each entry β_i of β contains a list of pair (b, ν_a) , where b is the length of the border and ν_a represents the required assignment.
- This list is kept sorted in decreasing order of border lengths of $x[1 \dots i]$.





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The First Step: Running Time Analysis

- If we assume that the maximum number of borders of any prefix of x is m, then the worst case running time of the algorithm is O(nm).
- But from Lemma 3 we know that the expected number of borders of an indeterminate string is bounded by a constant.
- As a result the expected running time of the above algorithm is O(n).





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The Second Step

- Now, we find out the covers of string x. Here we need only the last entry of the border array, β_n , where n = |x|.
- To identify a border as a cover of x we use the pattern matching technique of an Aho-Corasick automaton.
- We build an Aho-Corasick automaton with the dictionary containing the border of x and parse x through the automaton to find out whether x can be covered by the it or not.





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The Second Step: Building the Aho-Corasick automaton

- Suppose in iteration i, we have the length of the ith border of β_n equal to b.
- In this iteration, we build an Aho-Corasick automaton for the following dictionary:

$$D = \{x[1]x[2]...x[b]\},$$
 where $\forall j \in 1$ to b, $x[j] \in \Sigma$





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The Second Step: Function isCover

Algorithm 1 formally presents the steps of a function *isCover()*, which is the heart of the second step.

- 1: Construct the Aho-Corasick automaton for c
- 2: parse *x* and compute the positions where *c* occurs in *x* and put the positions in the array *Pos*
- 3: **for** i = 2 to |Pos| **do**
- 4: **if** Pos[i] Pos[i-1] > |c| **then**
- 5: Return FALSE
- 6: end if
- 7: end for
- 8: Return TRUE

Algorithm 1: Function isCover(x, c)





The Second Step: Running Time Analysis of Algorithm 1

- Clearly, Steps 3 and 2 run in O(n).
- Now, the complexity of Step 1 is linear in the size of the dictionary on which the automaton is build.
- Here the length of the string in the dictionary can be n-1 in the worst case. So, the time and space complexity of this algorithm is O(n).





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- According to Fact 2, if u and c are covers of x and |u| < |c| then u must be a cover of c.
- Now if $\beta_n = \{b_1, b_2, \dots, b_m\}$ then from the definition of border array $b_1 > b_2 > \dots > b_n$.
- Now if in any iteration we find a b_i that is a cover of x then from Fact 2, we can say that for all j ∈ i + 1 ... m, if b_j is a cover of x if and only if b_j is a cover of b_i.
- So instead of parsing x we can parse b_i for the subsequent automatons and as $|b_i| \le |x|$.





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The Second Step: Overall Algorithms

```
    k ← n
    AC ← φ {AC is a list used to store the covers of x}
    for all b ∈ β<sub>n</sub> do
    if isCover(x[1..k], x[1..b]) = true then
    m ← b
    AC.Add(k)
    end if
    end for
    Algorithm 2: Computing all covers of x
```





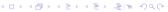
- The running time of Algorithm 2 is O(nm), where m is number of borders of x or alternatively number of entries in β_n .
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- It follows from above that our algorithm for finding all the covers of an indeterminate string of length n runs in O(n) time on the average.
- The worst case complexity of our algorithm is O(nm), i.e., $O(n^2)$.
- Which is also an improvement since the current best known algorithm [2] for finding all covers requires O(n² log n) in the worst case.





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Input: We are given an indeterminate string x, of length n on a fixed alphabet Σ .

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Algorithm for Computing the Cover Array

- Here we only need the length of the largest border of each prefix of x. This information is stored in the first entry of each β_i of the border array.
- Let us assume that $\beta_i[1]$ denotes the first entry of the list β_i that is $\beta_i[1]$ is the length of the largest border of x[1...i].





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Algorithm for Computing the Cover Array

```
1: \gamma[i] \leftarrow 0 \quad \forall i \in \{1 \dots n\}

2: for i \leftarrow 1 to n do

3: if isCover(x, x[1 \dots \beta_{i[1]}]) = true then

4: \gamma[i] \leftarrow \beta_{i[1]}

5: end if

6: end for

Algorithm 3: Computing cover array \gamma of x
```





Running Time Analysis

• As the worst case running time of the isCover(x, c) function is O(n) and the algorithm iterates over the n lists of the border array β , the running time of Algorithm 3 is $O(n^2)$.





Summary

- In this paper we have presented an average case O(n) time and space complex algorithm for computing all the covers of a given indeterminate string x of length n.
- We have also presented an algorithm for computing the cover array γ of an indeterminate string. This algorithm requires $O(n^2)$ time and O(n) space in the worst case.
- Both of these algorithms are improvement over existing algorithms.





Thank You





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Appendix

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