# Asymptotic Behaviour of the Maximal Number of Squares in Standard Sturmian Words

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# Sturmian words

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- Types of squares
- Direct formula for the number of squares
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Squares vs. runs

Definition Example

#### Sturmian word

- Alphabet:  $\Sigma = \{a, b\}$
- Directive sequence:  $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ , where  $\gamma_0 \ge 0$  and  $\gamma_1, \dots, \gamma_n > 0$

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Recurrence:

• 
$$x_{-1} = b$$

• 
$$x_0 = a$$
  
•  $x_k = (x_{k-1})^{\gamma_{k-1}} \cdot x_k$ 

• 
$$Sw(\gamma_0, \gamma_1, \ldots, \gamma_n) = x_{n+1}$$

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## Sturmian word example

- $\gamma = (1, 2, 1, 3, 1)$  $x_1 = b$ •  $x_0 = a$ •  $x_1 = (x_0)^1 \cdot x_{-1} = a \cdot b$ •  $x_2 = (x_1)^2 \cdot x_0 = ab \cdot ab \cdot a$ •  $x_3 = (x_2)^1 \cdot x_1 = ababa \cdot ab$ •  $x_4 = (x_3)^3 \cdot x_2 = ababaab \cdot ababaab \cdot ababaab \cdot ababaab$



## Fibonacci word example

- $\gamma = (1, 1, 1, 1, 1)$
- $x_{-1} = b$
- *x*<sub>0</sub> = *a*
- $x_1 = x_0 \cdot x_{-1} = a \cdot b$
- $x_2 = x_1 \cdot x_0 = ab \cdot a$
- $x_3 = x_2 \cdot x_1 = aba \cdot ab$
- $x_4 = x_3 \cdot x_2 = abaab \cdot aba$
- $x_5 = x_4 \cdot x_3 = abaababa \cdot abaab$



Basics Fibonacci words





Notation:

- sq(w) the number of distinct squares in the word w
- $\rho(w)$  the number of runs in the word w



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Basics Fibonacci words

## Squares

$$sq(F_n) = 2|F_{n-2}| - 2$$

#### Runs

$$\rho(F_n) = 2|F_{n-2}| - 3$$

$$\frac{\rho(F_n)}{|F_n|} \longrightarrow 0.763932\ldots$$

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Types of squares Direct formula for the number of squares Asymptotic behaviour of the number of squares Squares vs. runs

#### Lemma (Damanik, Lenz 2003)

Each primitive period of a square in  $Sw(\gamma_0, \ldots, \gamma_n)$  has the length  $\mathbf{k}|\mathbf{x}_i|$  for  $1 \le k \le \gamma_i$  or  $\mathbf{k}|\mathbf{x}_i| + |\mathbf{x}_{i-1}|$  for  $1 \le k < \gamma_i$ .

#### Notation:

- squares with period of the length  $\mathbf{k}|\mathbf{x}_i|$  for  $1 \le k \le \gamma_i$  or  $\mathbf{k}|\mathbf{x}_i| + |\mathbf{x}_{i-1}|$  for  $1 \le k < \gamma_i$  are said to be of type i
- squares with period of the form  $a^n$  are said to be of type 0

Sturmian words Repetitions The number of squares Squares v, runs



• 
$$w = Sw(3, 3, 2, 1, 1)$$
  
•  $|x_0| = 1, |x_1| = 4, |x_2| = 13, |x_3| = 30, |x_4| = 43$ 

• 
$$sq(w) = 53$$

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#### Number of squares

$$sq(w) = \sum_{i=0}^{n} sq_n(w)$$

## • $sq_i(w)$ - the number of squares of type i in the word w



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$$\mathbf{w} = \mathbf{Sw}(\gamma_0, \gamma_1, \dots, \gamma_n)$$

## Type i – general case (0 < i < n - 2)

$$\mathsf{sq}_i(w) = \left\{ egin{array}{c} \left(rac{3}{2}\gamma_i - 1
ight) |x_i| + |x_{i-1}| - \gamma_i & ext{even } \gamma_i \ \left(rac{3}{2}\gamma_i - rac{1}{2}
ight) |x_i| - \gamma_i + 1 & ext{odd } \gamma_i \end{array} 
ight.$$



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Type n-2 
$$(\gamma_n > 1)$$

$$sq_{n-2}(w) = \begin{cases} \left(\frac{3}{2}\gamma_{n-2} - 1\right)|x_{n-2}| + |x_{n-3}| - \gamma_{n-2} & \text{even } \gamma_{n-2} \\ \left(\frac{3}{2}\gamma_{n-2} - \frac{1}{2}\right)|x_{n-2}| - \gamma_{n-2} + 1 & \text{odd } \gamma_{n-2} \end{cases}$$

# Type n-2 ( $\gamma_n=1$ )

$$sq_{n-2}(w) = \begin{cases} \left(\frac{3}{2}\gamma_{n-2} - 1\right)|x_{n-2}| - \gamma_{n-2} + 1 & \text{even } \gamma_{n-2} \\ \left(\frac{3}{2}\gamma_{n-2} - \frac{3}{2}\right)|x_{n-2}| + |x_{n-3}| - \gamma_{n-2} + 2 & \text{odd } \gamma_{n-2} \end{cases}$$

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# Type n-1 $(\gamma_n > 1)$

$$sq_{n-1}(w) = \begin{cases} \left(\frac{3}{2}\gamma_{n-1} - 1\right)|x_{n-1}| - \gamma_{n-1} + 1 & \text{even } \gamma_{n-1} \\ \left(\frac{3}{2}\gamma_{n-1} - \frac{3}{2}\right)|x_{n-1}| + |x_{n-2}| - \gamma_{n-1} & \text{odd } \gamma_{n-1} \end{cases}$$

# Type n-1 ( $\gamma_n = 1$ )

$$sq_{n-1}(w) = \begin{cases} \frac{1}{2}\gamma_{n-1}|x_{n-1}| & \text{even } \gamma_{n-1} \\ \frac{1}{2}\gamma_{n-1} - \frac{1}{2}|x_{n-1}| + |x_{n-2}| - 1 & \text{odd } \gamma_{n-1} \end{cases}$$

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# Type n

$$sq_n(w) = \begin{cases} \left(\frac{1}{2}\gamma_n - 1\right)|x_n| + |x_{n-1}| + 1 & \text{even } \gamma_n \\ \\ \left(\frac{1}{2}\gamma_n - \frac{1}{2}\right)|x_n| & \text{odd } \gamma_n \end{cases}$$

## Type 0

$$sq_0(w) = \left\lfloor \frac{\gamma_0 + 1}{2} \right\rfloor$$

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## How to compute $|x_i|$ ?

• 
$$|x_{-1}| = 1$$

• 
$$|x_0| = 1$$

. . .

• 
$$|x_1| = \gamma_0 |x_0| + |x_{-1}|$$

• 
$$|x_2| = \gamma_1 |x_1| + |x_0|$$

• 
$$|x_{n+1}| = \gamma_n |x_n| + |x_{n-1}|$$



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## Example:

• 
$$w = Sw(3, 3, 2, 1, 1)$$

• 
$$sq_0(w) = \lfloor \frac{3+1}{2} \rfloor = 2$$
  
•  $sq_1(w) = \frac{1}{2}(3 \cdot 3^2 + 1) = 14$ 

• 
$$sq_2(w) = 2 \cdot 3^2 + 2 \cdot 3 + 1 = 25$$

• 
$$sq_3(w) = 3^2 + 3 = 12$$

•  $sq_4(w) = 0$ 

• 
$$sq(w) = 2 + 14 + 25 + 12 + 0 = 53$$

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#### Theorem

$$sq(w) \leq 0.9|w| + 4$$



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$$w_{\mathbf{k}} = \mathbf{Sw}(\mathbf{k}, \mathbf{k}, 2, 1, 1)$$
  
•  $w_{k} = ((a^{k}b)^{k}a)^{2}a^{k}b((a^{k}b)^{k}a)^{3}a^{k}b$ 

• 
$$|w_k| = 5k^2 + 7k + 7$$

• 
$$sq(w_k) = \begin{cases} \frac{1}{2} \left(9k^2 + 6k + 2\right) & \text{for even } k \\ \frac{1}{2} \left(9k^2 + 7k + 4\right) & \text{for odd } k \end{cases}$$
  
•  $\frac{sq(w_k)}{|w_k|} \longrightarrow 0.9$ 

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## Theorem (Baturo, Piątkowski, Rytter 2008)

$$\frac{\rho(w)}{|w|} \longrightarrow 0.8$$



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$$\mathsf{w}_{\mathsf{k}}=\mathsf{Sw}(\mathsf{k},\mathsf{k},2,1,1)$$

• 
$$|w_k| = 5k^2 + 7k + 7$$

• 
$$sq(w_k) = \begin{cases} \frac{1}{2} \left(9k^2 + 6k + 2\right) & \text{for even } k \\ \frac{1}{2} \left(9k^2 + 7k + 4\right) & \text{for odd } k \end{cases}$$
  
•  $\frac{sq(w_k)}{|w_k|} \longrightarrow 0.9$ 

• 
$$\rho(w_k) = 9k + 7$$
  
•  $\frac{\rho(w_k)}{|w_k|} \longrightarrow 0$ 

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$$\mathbf{v_k} = \mathbf{Sw}(\mathbf{1},\mathbf{2},\mathbf{k},\mathbf{k})$$

• 
$$|v_k| = 5k^2 + 2k + 5$$

• 
$$\rho(v_k) = 4k^2 - k + 3$$
  
•  $\frac{\rho(v_k)}{|v_k|} \longrightarrow 0.8$ 

•  $\frac{sq(v_k)}{|v_k|} \longrightarrow 0.5$ 

• 
$$sq(v_k) = \begin{cases} \frac{1}{2} \left(5k^2 + 5k + 8\right) & \text{for even } k \\ \frac{1}{2} \left(5k^2 + 10k - 5\right) & \text{for odd } k \end{cases}$$

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