Asymptotic Behaviour of the Maximal Number of Squares in Standard Sturmian Words

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Sturmian word

- Alphabet: $\Sigma = \{a, b\}$
- Directive sequence: $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$, where $\gamma_0 \geq 0$ and $\gamma_1, \ldots, \gamma_n > 0$
- Recurrence:
  - $x_{-1} = b$
  - $x_0 = a$
  - $x_k = (x_{k-1})^{\gamma_{k-1}} \cdot x_{k-2}$
- $\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n) = x_{n+1}$

Sturmian word example

- $\gamma = (1, 2, 1, 3, 1)$

- $x_{-1} = b$
- $x_0 = a$
- $x_1 = (x_0)^1 \cdot x_{-1} = a \cdot b$
- $x_2 = (x_1)^2 \cdot x_0 = ab \cdot ab \cdot a$
- $x_3 = (x_2)^1 \cdot x_1 = ababa \cdot ab$
- $x_4 = (x_3)^3 \cdot x_2 = ababaab \cdot ababaab \cdot ababaab \cdot ababa$
- $x_5 = (x_4)^1 \cdot x_3 = ababaababaababaababaababa \cdot ababaab$

- $\mathrm{Sw}(1, 2, 1, 3, 1) = ababaababaababaababaababaababaab$

Fibonacci word example

- $\gamma = (1,1,1,1,1)$

- $x_{-1} = b$
- $x_0 = a$
- $x_1 = x_0 \cdot x_{-1} = a \cdot b$
- $x_2 = x_1 \cdot x_0 = ab \cdot a$
- $x_3 = x_2 \cdot x_1 = aba \cdot ab$
- $x_4 = x_3 \cdot x_2 = abaab \cdot aba$
- $x_5 = x_4 \cdot x_3 = abaababa \cdot abaab$

- $\mathrm{Sw}(1,1,1,1,1) = abaababaabaab$

Square

$$\text{aba}\,\underbrace{\text{baababa}}\,\underbrace{\text{baababa}}\,\text{baababaabababaab}$$

Run

$$\underbrace{\text{ababaab}}\,\underbrace{\text{ababaab}}\,\underbrace{\text{ababaab}}\,\underbrace{\text{ababaab}}\,\underbrace{\text{aba}}\,\text{ab}$$

Notation:

- $sq(w)$ – the number of distinct squares in the word w
- $\rho(w)$ – the number of runs in the word w

Squares

$$sq(F_n) = 2|F_{n-2}| - 2$$

Runs

$$\rho(F_n) = 2|F_{n-2}| - 3$$

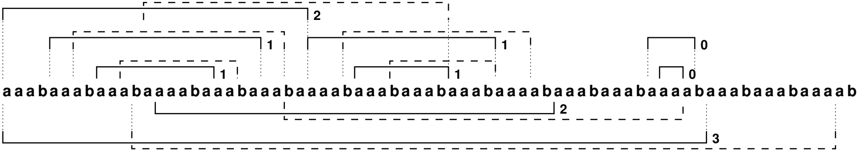
$$\frac{\rho(F_n)}{|F_n|} \longrightarrow 0.763932\ldots$$

Lemma (Damanik, Lenz 2003)

Each primitive period of a square in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ has the length $\mathbf{k}|\mathbf{x_i}|$ for $1 \leq k \leq \gamma_i$ or $\mathbf{k}|\mathbf{x_i}| + |\mathbf{x_{i-1}}|$ for $1 \leq k < \gamma_i$.

Notation:

- squares with period of the length $\mathbf{k}|\mathbf{x_i}|$ for $1 \leq k \leq \gamma_i$ or $\mathbf{k}|\mathbf{x_i}| + |\mathbf{x_{i-1}}|$ for $1 \leq k < \gamma_i$ are said to be of **type i**
- squares with period of the form a^n are said to be of **type 0**



- $w = \mathrm{Sw}(3, 3, 2, 1, 1)$
- $|x_0| = 1,\ |x_1| = 4,\ |x_2| = 13,\ |x_3| = 30,\ |x_4| = 43$

- type 0 – 2 $(1 \cdot \mathbf{1} \ + \ 1 \cdot \mathbf{2})$
- type 1 – 14 $(4 \cdot \mathbf{4} \ + \ 3 \cdot \mathbf{5} \ + \ 4 \cdot \mathbf{8} \ + \ 3 \cdot \mathbf{9})$
- type 2 – 25 $(13 \cdot \mathbf{13} \ + \ 12 \cdot \mathbf{17})$
- type 3 – 12 $(12 \cdot \mathbf{30})$

- $sq(w) = 53$

Number of squares

$$sq(w) = \sum_{i=0}^{n} sq_n(w)$$

- $sq_i(w)$ – the number of squares of **type i** in the word w

$$\mathbf{w} = \mathbf{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n)$$

Type i – general case ($0 < i < n-2$)

$$sq_i(w) = \begin{cases} \left(\frac{3}{2}\gamma_i - 1\right)|x_i| + |x_{i-1}| - \gamma_i & \text{even } \gamma_i \\ \left(\frac{3}{2}\gamma_i - \frac{1}{2}\right)|x_i| - \gamma_i + 1 & \text{odd } \gamma_i \end{cases}$$

Type n-2 ($\gamma_n > 1$)

$$sq_{n-2}(w) = \begin{cases} \left(\frac{3}{2}\gamma_{n-2} - 1\right)|x_{n-2}| + |x_{n-3}| - \gamma_{n-2} & \text{even } \gamma_{n-2} \\ \left(\frac{3}{2}\gamma_{n-2} - \frac{1}{2}\right)|x_{n-2}| - \gamma_{n-2} + 1 & \text{odd } \gamma_{n-2} \end{cases}$$

Type n-2 ($\gamma_n = 1$)

$$sq_{n-2}(w) = \begin{cases} \left(\frac{3}{2}\gamma_{n-2} - 1\right)|x_{n-2}| - \gamma_{n-2} + 1 & \text{even } \gamma_{n-2} \\ \left(\frac{3}{2}\gamma_{n-2} - \frac{3}{2}\right)|x_{n-2}| + |x_{n-3}| - \gamma_{n-2} + 2 & \text{odd } \gamma_{n-2} \end{cases}$$

Type n-1 ($\gamma_n > 1$)

$$sq_{n-1}(w) = \begin{cases} \left(\frac{3}{2}\gamma_{n-1} - 1\right)|x_{n-1}| - \gamma_{n-1} + 1 & \text{even } \gamma_{n-1} \\ \left(\frac{3}{2}\gamma_{n-1} - \frac{3}{2}\right)|x_{n-1}| + |x_{n-2}| - \gamma_{n-1} & \text{odd } \gamma_{n-1} \end{cases}$$

Type n-1 ($\gamma_n = 1$)

$$sq_{n-1}(w) = \begin{cases} \frac{1}{2}\gamma_{n-1}|x_{n-1}| & \text{even } \gamma_{n-1} \\ \left(\frac{1}{2}\gamma_{n-1} - \frac{1}{2}\right)|x_{n-1}| + |x_{n-2}| - 1 & \text{odd } \gamma_{n-1} \end{cases}$$

Type n

$$sq_n(w) = \begin{cases} \left(\frac{1}{2}\gamma_n - 1\right)|x_n| + |x_{n-1}| + 1 & \text{even } \gamma_n \\ \left(\frac{1}{2}\gamma_n - \frac{1}{2}\right)|x_n| & \text{odd } \gamma_n \end{cases}$$

Type 0

$$sq_0(w) = \left\lfloor \frac{\gamma_0 + 1}{2} \right\rfloor$$

How to compute $|\mathbf{x_i}|$?

- $|x_{-1}| \;=\; 1$
- $|x_0| \;=\; 1$
- $|x_1| \;=\; \gamma_0|x_0| + |x_{-1}|$
- $|x_2| \;=\; \gamma_1|x_1| + |x_0|$

  ...
- $|x_{n+1}| \;=\; \gamma_n|x_n| + |x_{n-1}|$

Example:

- $w = \mathrm{Sw}(3,3,2,1,1)$

- $sq_0(w) = \left\lfloor \frac{3+1}{2} \right\rfloor = \mathbf{2}$
- $sq_1(w) = \frac{1}{2}\left(3 \cdot 3^2 + 1\right) = \mathbf{14}$
- $sq_2(w) = 2 \cdot 3^2 + 2 \cdot 3 + 1 = \mathbf{25}$
- $sq_3(w) = 3^2 + 3 = \mathbf{12}$
- $sq_4(w) = \mathbf{0}$

- $sq(w) = 2 + 14 + 25 + 12 + 0 = \mathbf{53}$

Theorem

$$sq(w) \leq 0.9|w| + 4$$

$\mathbf{w_k = Sw(k, k, 2, 1, 1)}$

- $w_k \ = \ \left((a^k b)^k a\right)^2 a^k b \left((a^k b)^k a\right)^3 a^k b$

- $|w_k| = 5k^2 + 7k + 7$

- $sq(w_k) = \begin{cases} \frac{1}{2}\left(9k^2 + 6k + 2\right) & \text{for even } k \\ \frac{1}{2}\left(9k^2 + 7k + 4\right) & \text{for odd } k \end{cases}$

- $\dfrac{sq(w_k)}{|w_k|} \longrightarrow 0.9$

Theorem (Baturo, Piątkowski, Rytter 2008)

$$\frac{\rho(w)}{|w|} \longrightarrow 0.8$$

$\mathbf{w_k = Sw(k, k, 2, 1, 1)}$

- $|w_k| = 5k^2 + 7k + 7$
- $sq(w_k) = \begin{cases} \frac{1}{2}\left(9k^2 + 6k + 2\right) & \text{for even } k \\ \frac{1}{2}\left(9k^2 + 7k + 4\right) & \text{for odd } k \end{cases}$
- $\frac{sq(w_k)}{|w_k|} \longrightarrow 0.9$
- $\rho(w_k) = 9k + 7$
- $\frac{\rho(w_k)}{|w_k|} \longrightarrow 0$

$\mathbf{v_k = Sw(1, 2, k, k)}$

- $|v_k| = 5k^2 + 2k + 5$
- $\rho(v_k) = 4k^2 - k + 3$
- $\frac{\rho(v_k)}{|v_k|} \longrightarrow 0.8$
- $sq(v_k) = \begin{cases} \frac{1}{2}\left(5k^2 + 5k + 8\right) & \text{for even } k \\ \frac{1}{2}\left(5k^2 + 10k - 5\right) & \text{for odd } k \end{cases}$
- $\frac{sq(v_k)}{|v_k|} \longrightarrow 0.5$

Thank You
For Your Attention