An input sensitive online algorithm for LCS computation

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LLCS is a dual of indel edit distance:

- $ed_{id}(A, B) = n + m - 2\text{LLCS}(A, B)$

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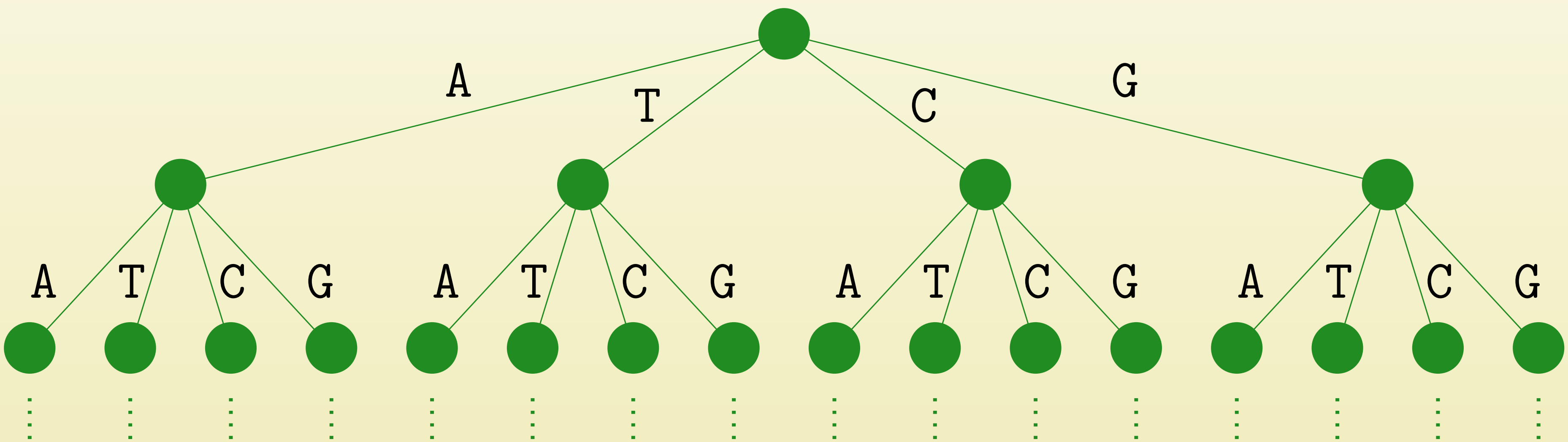

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Classic solution: $\mathcal{O}(mn)$ dynamic programming

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- A table D, where $D[i,j]$ =LLCS$(A_{1..i}, B_{1..j})$ and

$$D[i,j] = \begin{cases} 1 + D[i\text{-}1, j\text{-}1], \text{ if } A_i = B_j, \\ \text{else } \max\{D[i, j\text{-}1], D[i\text{-}1, j]\} \end{cases}$$

		c	h	a	r	g	e
	0	**0**	**0**	**0**	**0**	**0**	**0**
P	**0**	0	0	0	0	0	0
r	**0**	0	0	0	1	1	1
a	**0**	0	0	1	1	1	1
g	**0**	0	0	1	1	2	2
u	**0**	0	0	1	1	2	2
e	**0**	0	0	1	1	2	3

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		A	T	C
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- let $I_x[j]$ denote the xth increment point in column j

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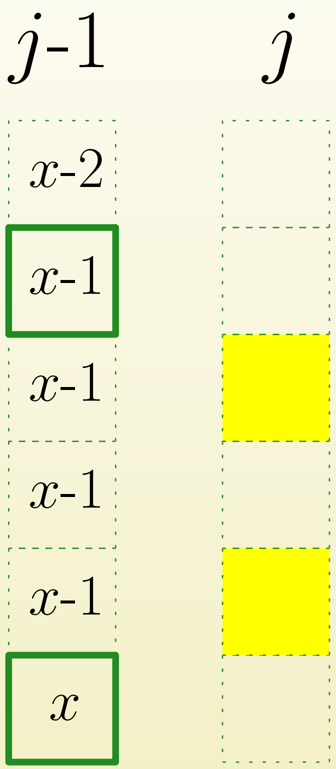

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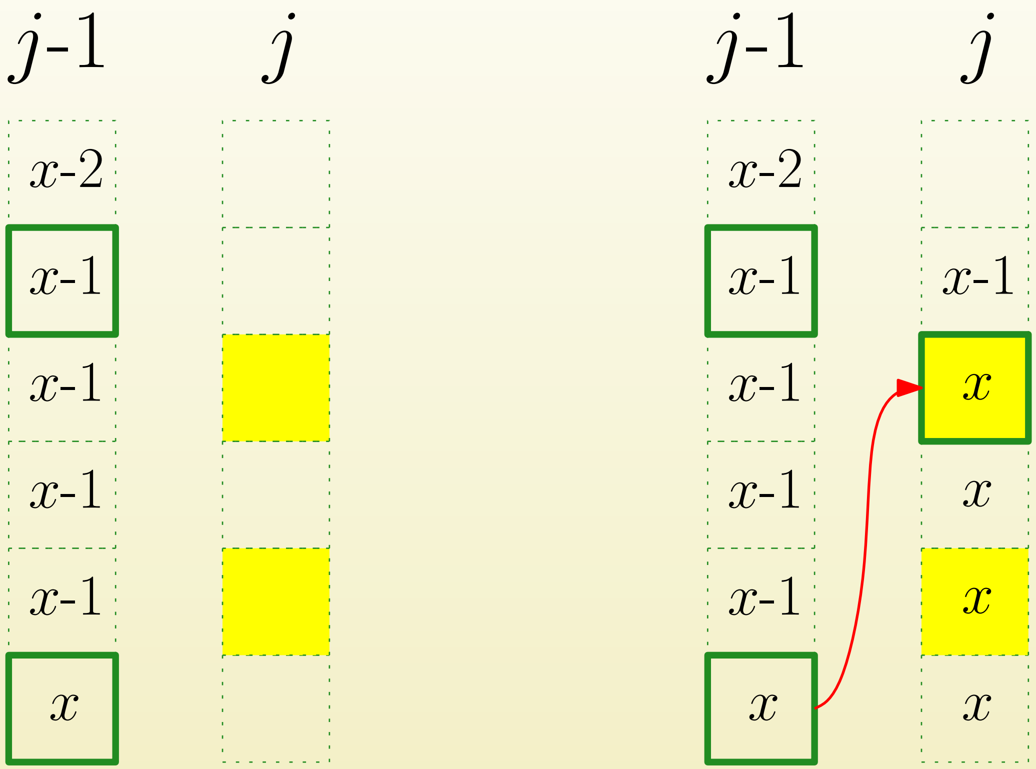

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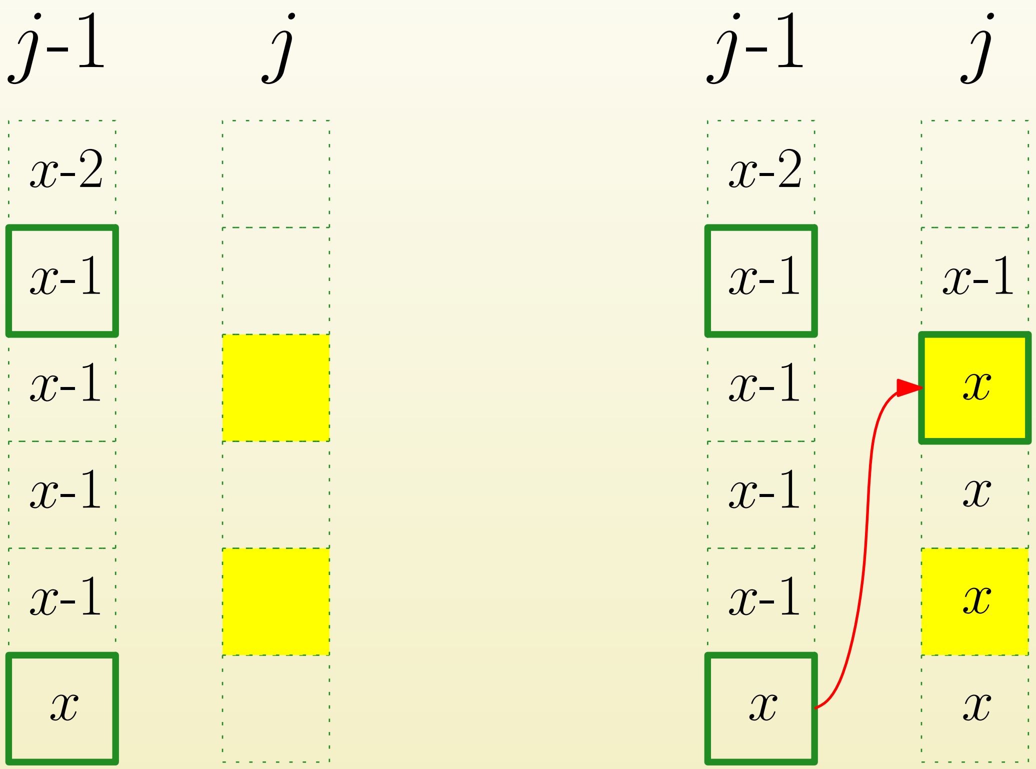

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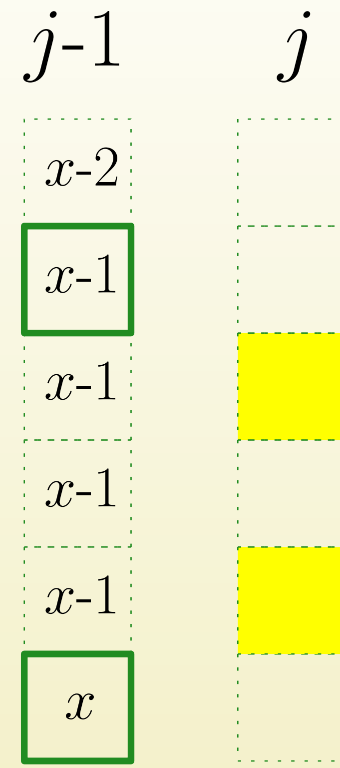
- Precompute a $\sigma \times n$ table NM, where $NM[\lambda, k] = \min\{i : i > k \wedge (A_i = \lambda \vee i = n+1)\}$



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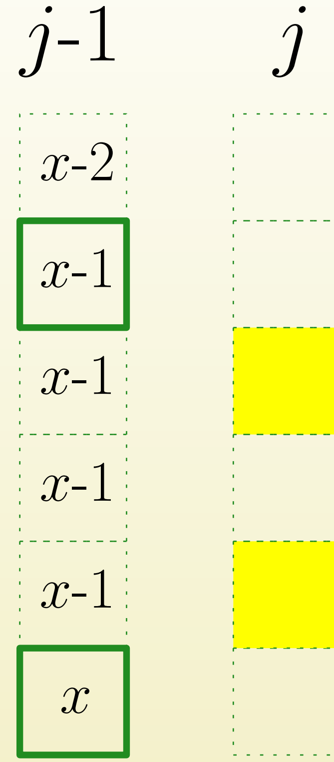

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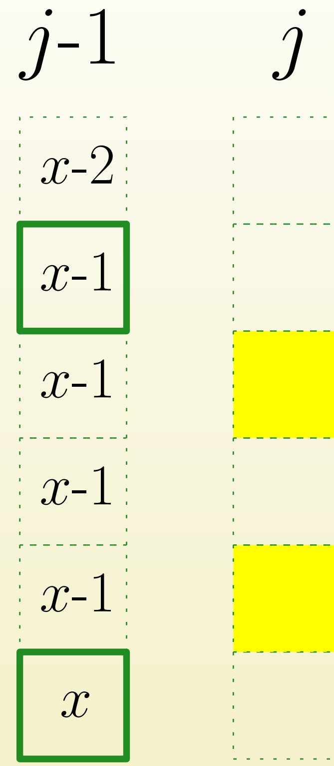

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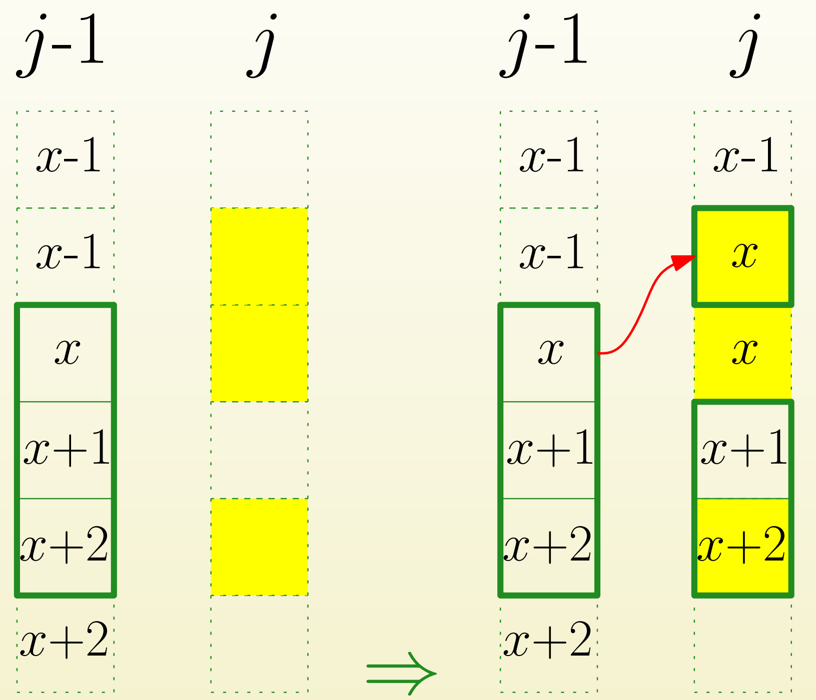
Resulting time to compute $L =$ LLCS(A, B): $\mathcal{O}(\sigma n + mL)$

Consider again the computation of the increment points

- The changes occur only to the first increment points among blocks of consecutive increments



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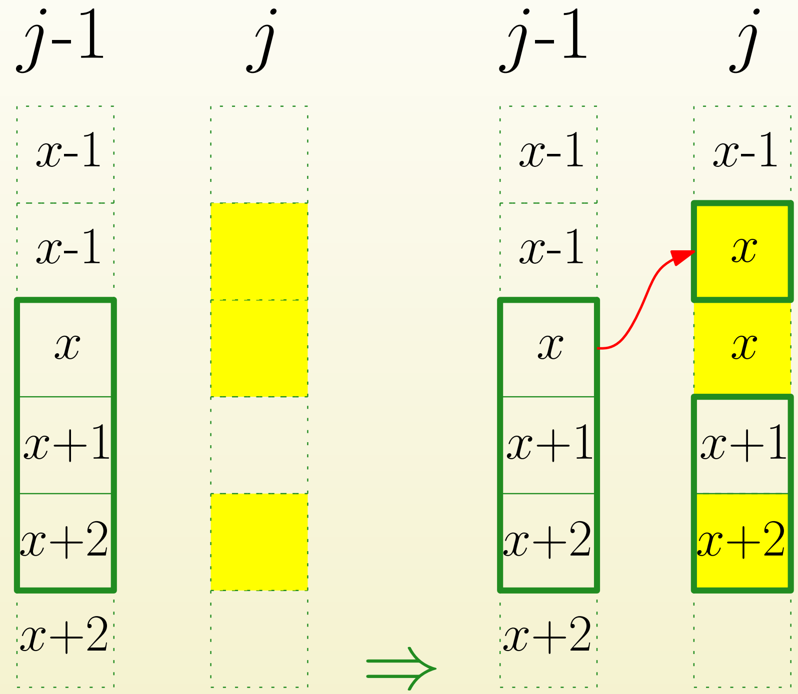

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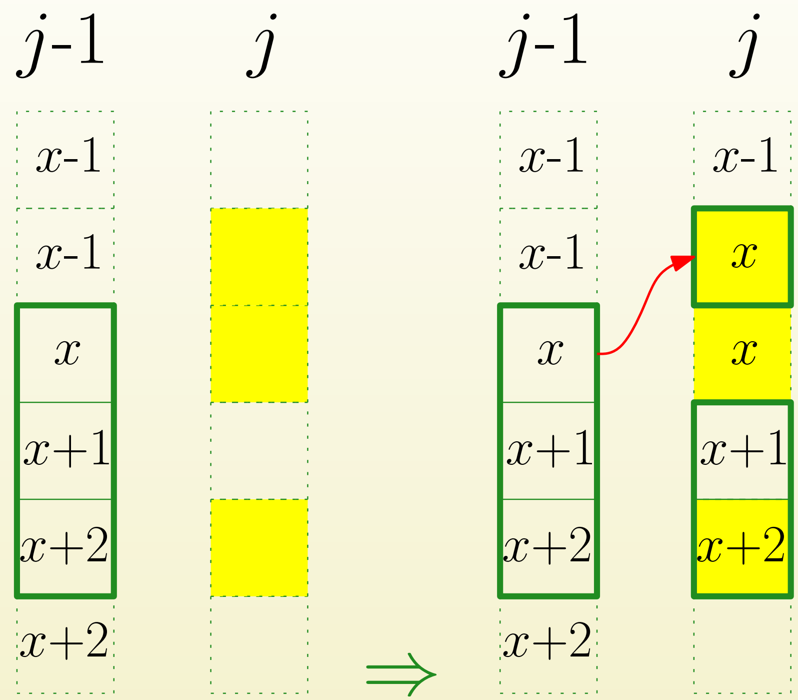

A "block" encoding: let $S_y[j]$ and $E_y[j]$ be the positions of the first and last points in the yth maximal segment of consecutive increment points

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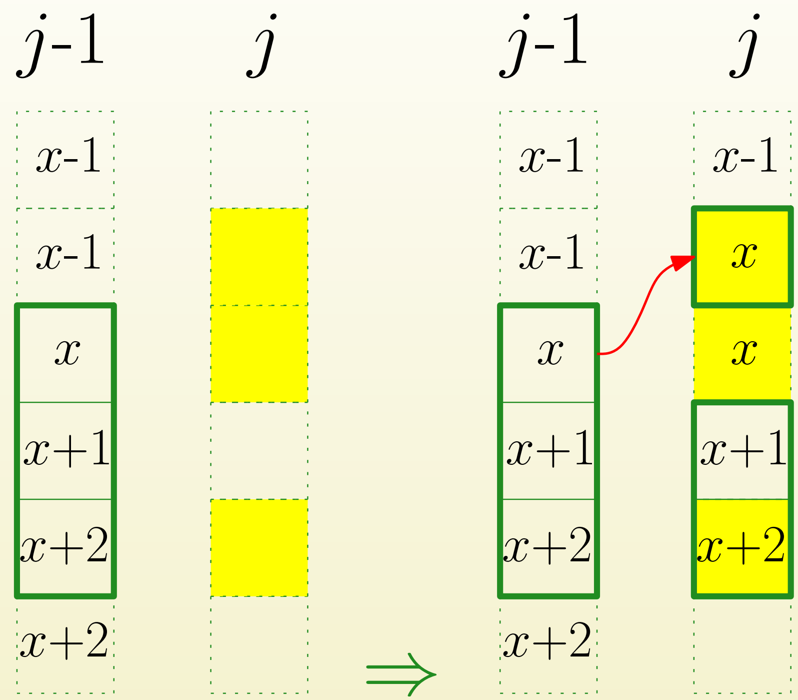

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- $\Delta[k, j] = 1$ for $k = S_y[j]..E_y[j]$
- $\Delta[k, j] \neq 0$ for $k = S_y[j]$-1 and $k = E_y[j] + 1$

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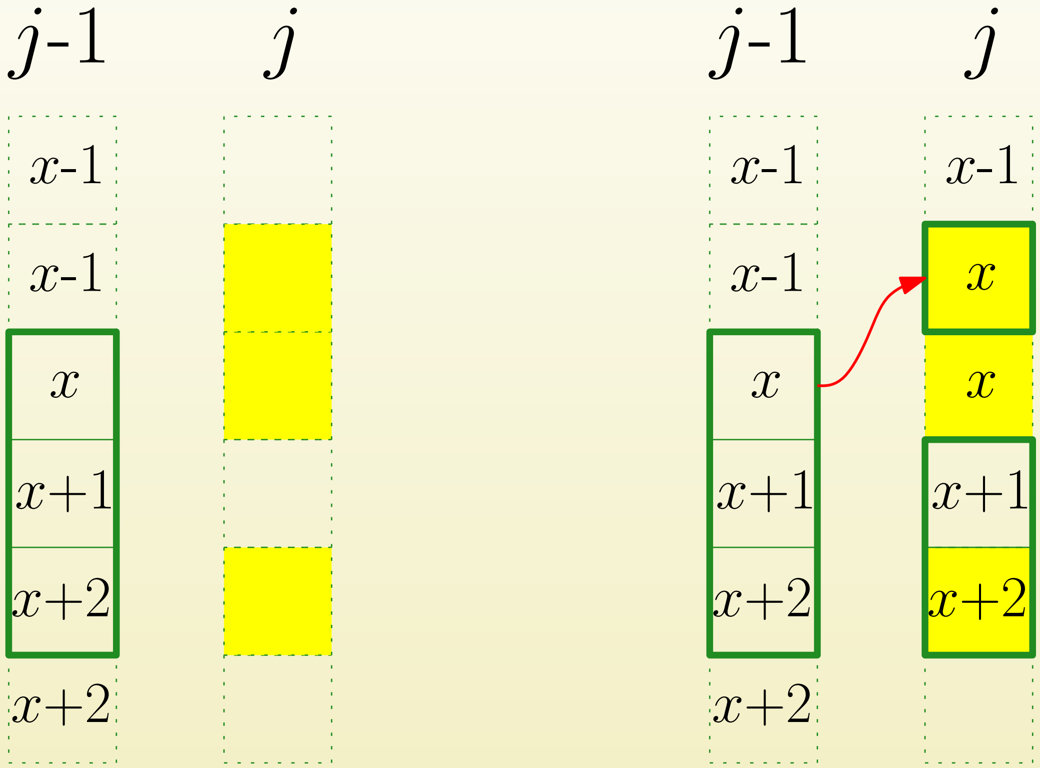

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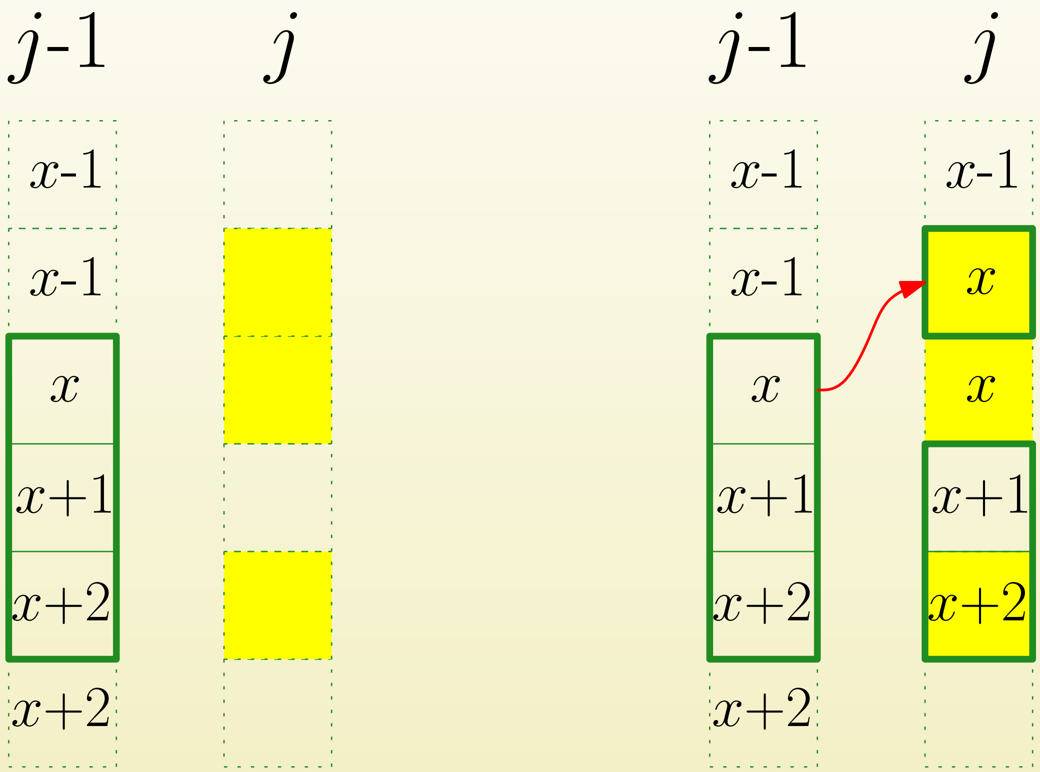

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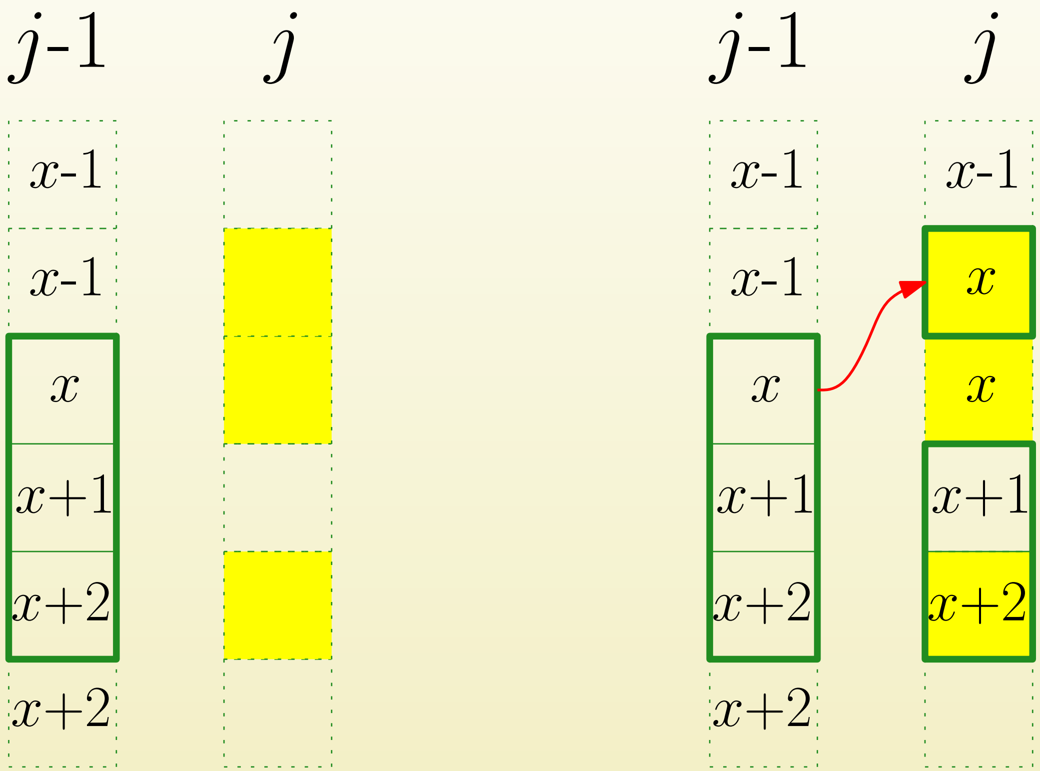

if $i = NM[B_j, E_{y-1}[j\text{-}1]] < S_y[j\text{-}1]$, add increment point i and increment block $S_y[j\text{-}1]+1..E_y[j\text{-}1]$ to column j (possibly merging point i into a block)

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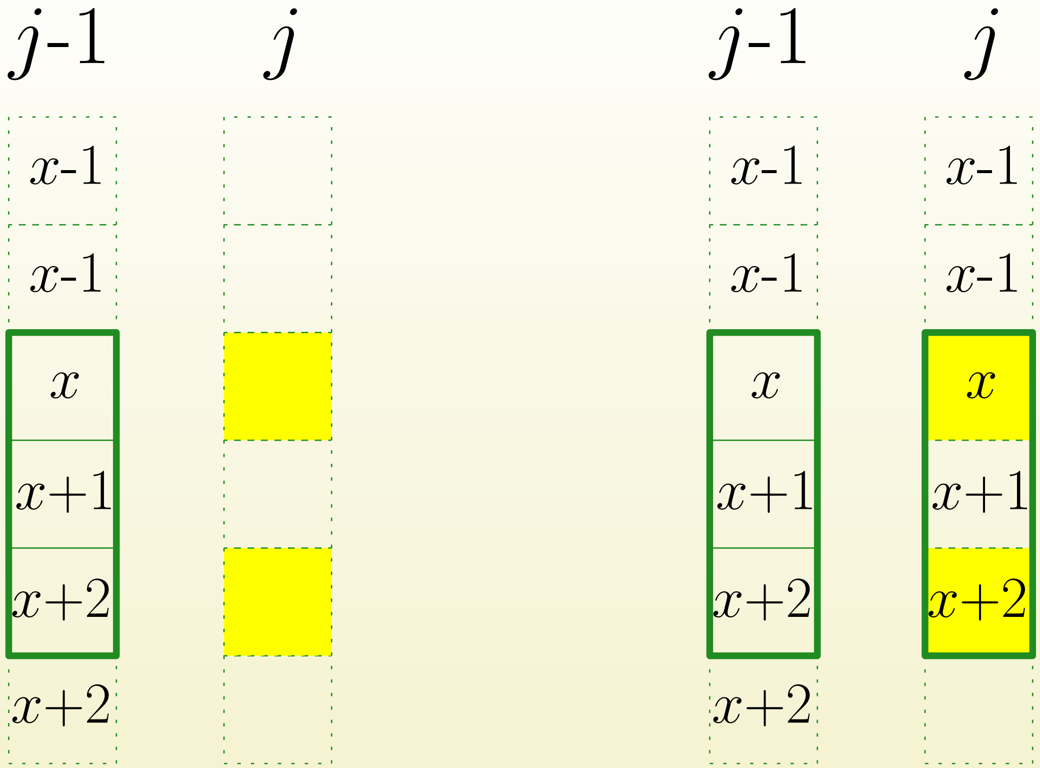

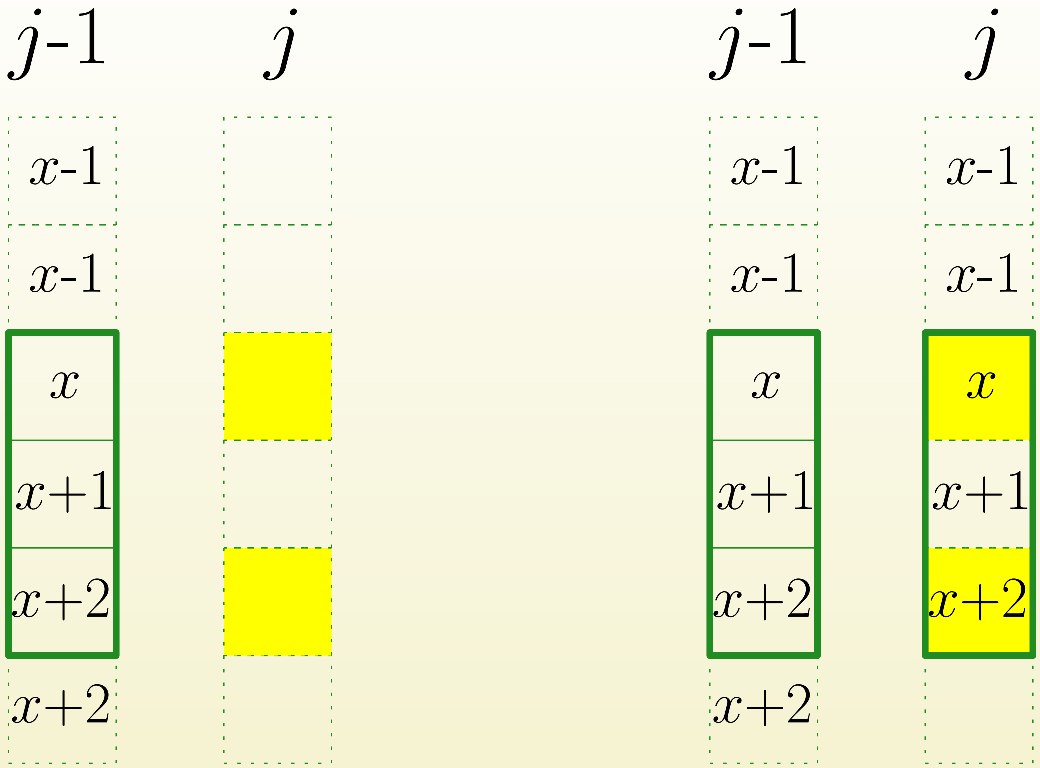

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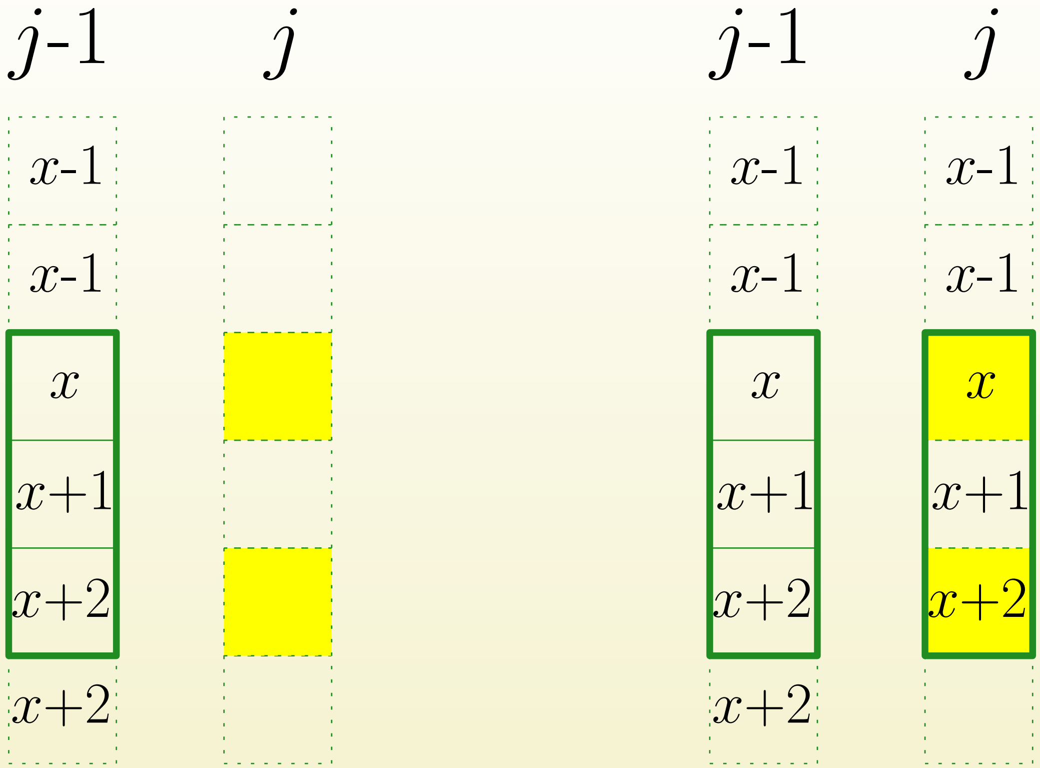

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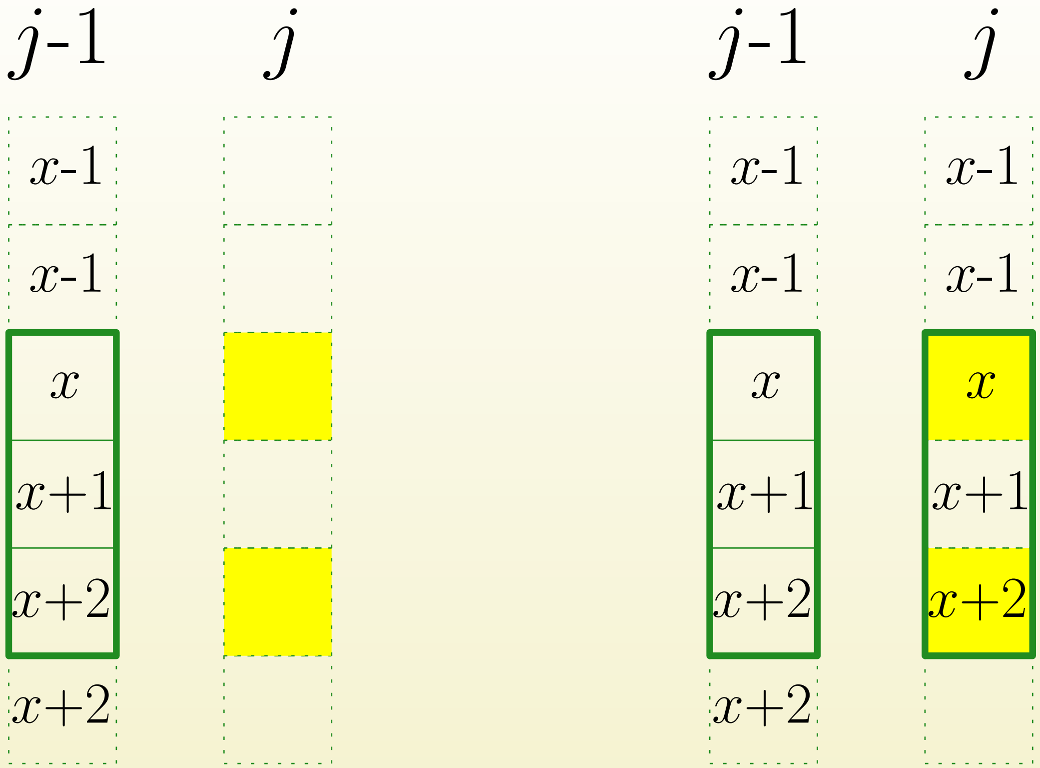

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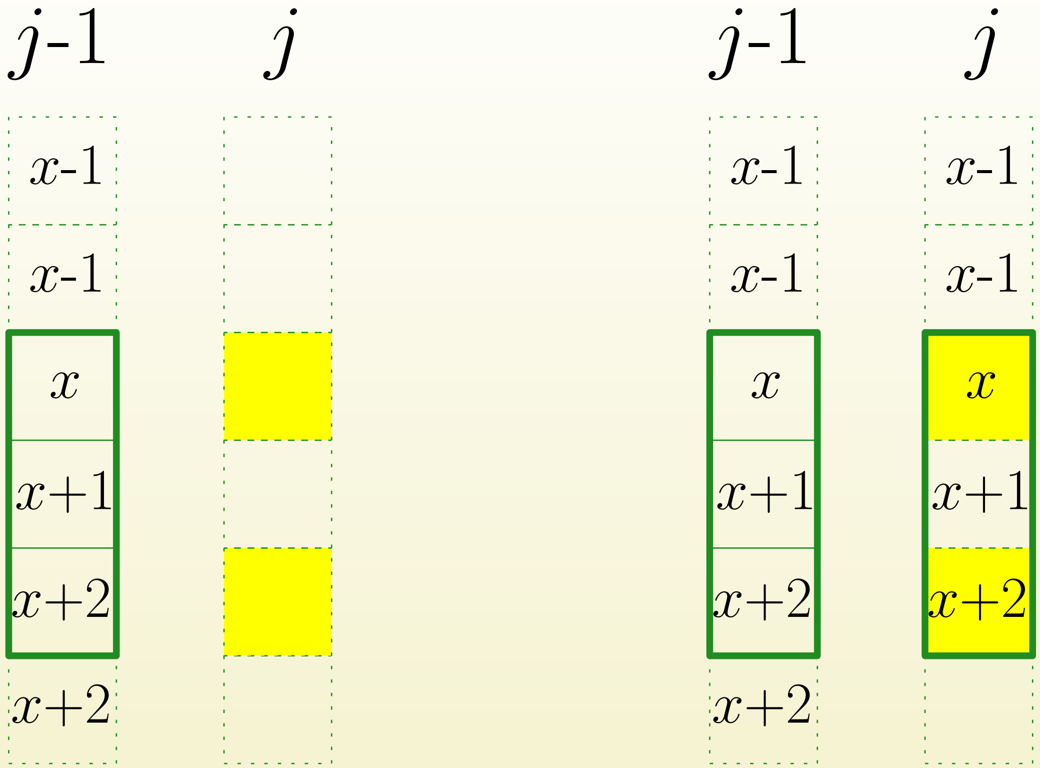
Amount of work



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Amount of work $\approx$ the number of increment blocks

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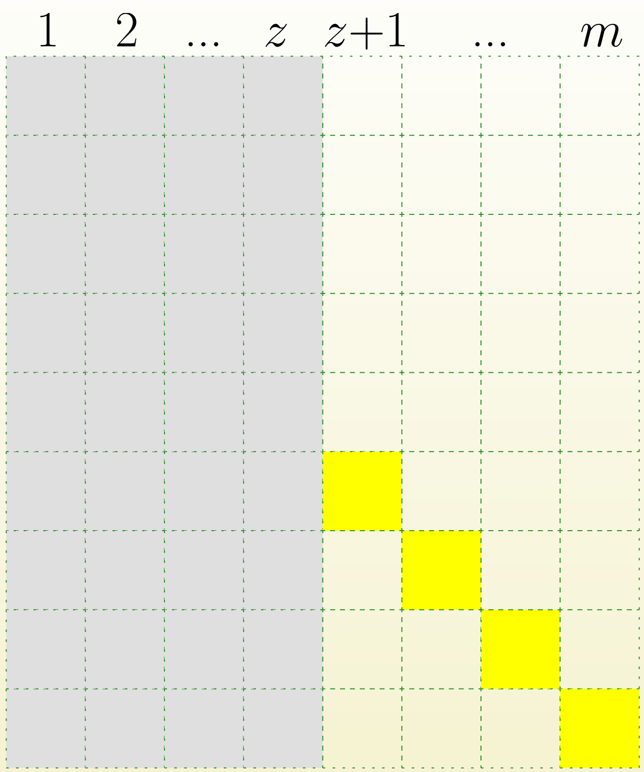
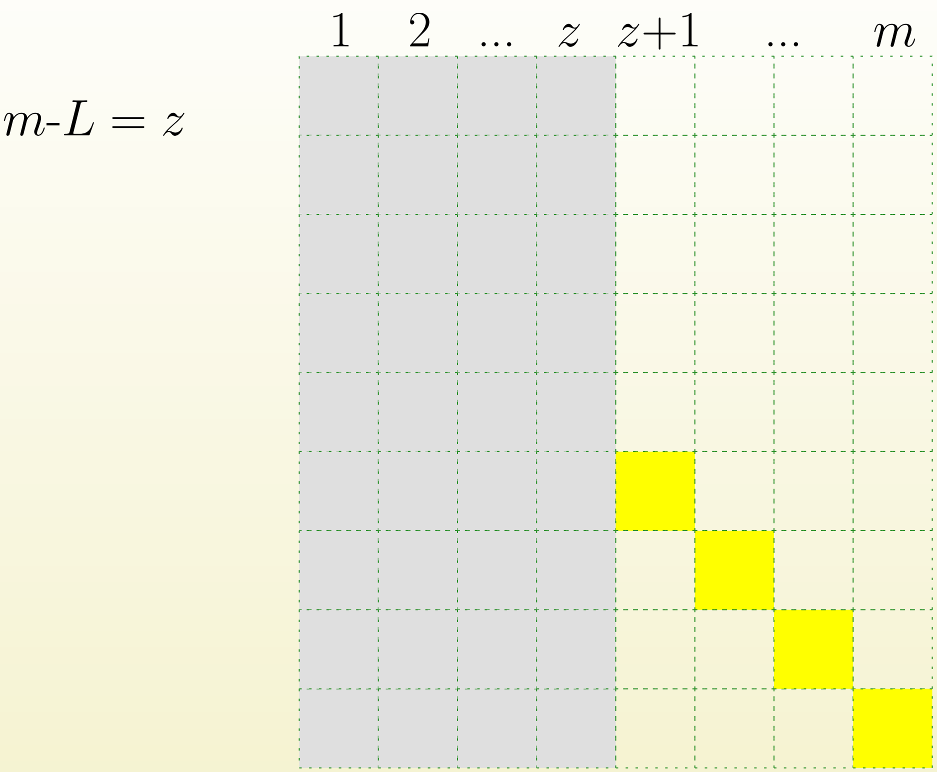
  - ▷ $block_\# \leq 1 + 0_\# = l - 1_\# + 1$

m-$L = z$



Consider the figure



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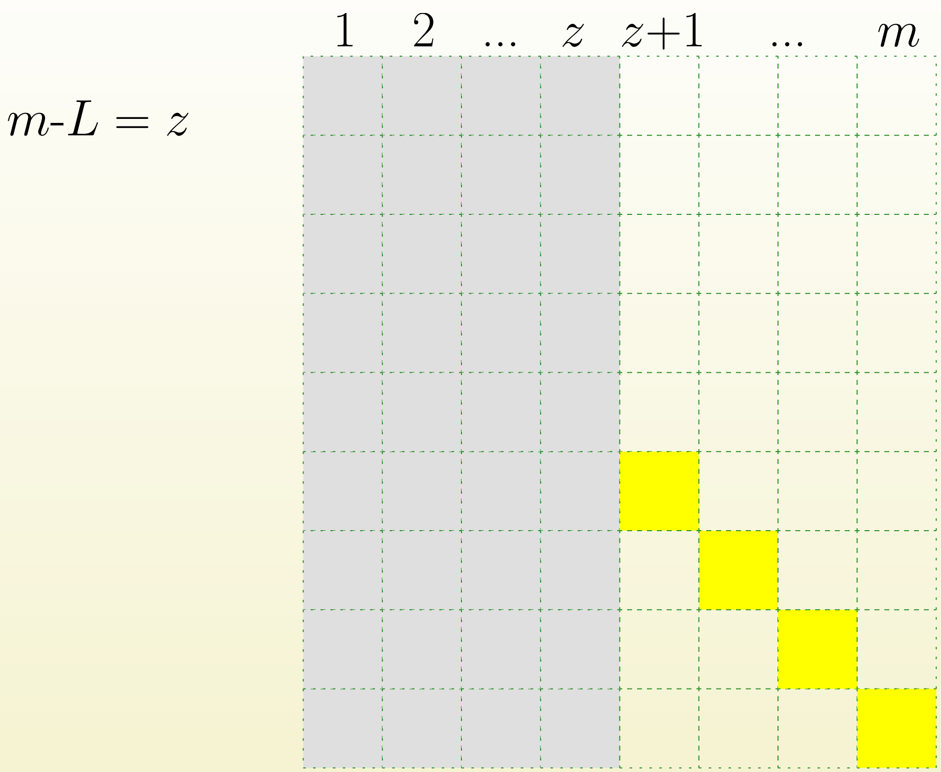

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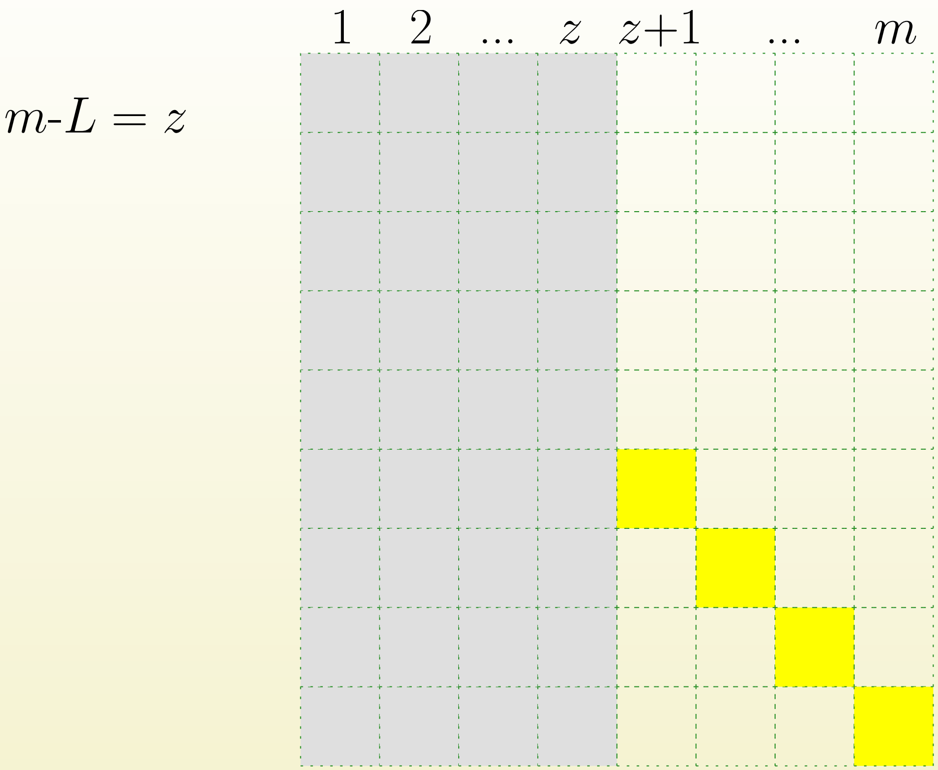

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  - ▷ Work for this part: $\mathcal{O}(z^2)$

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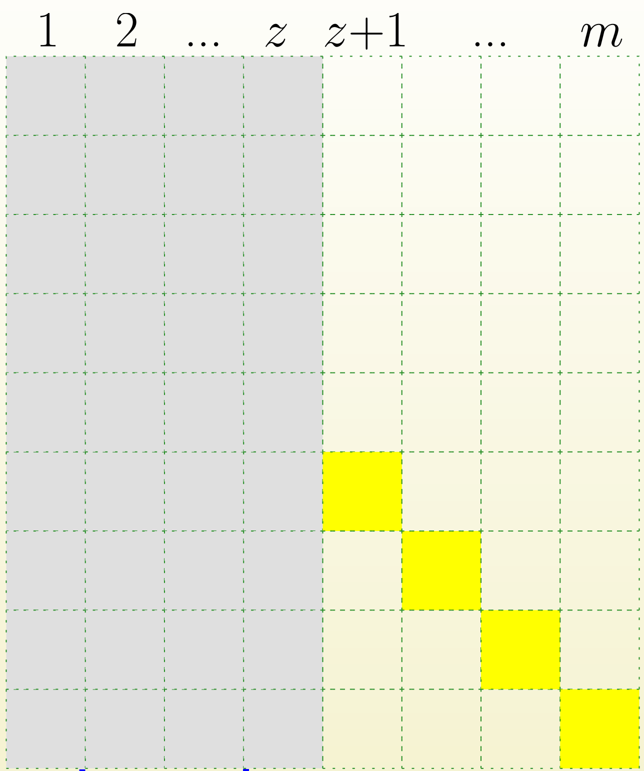

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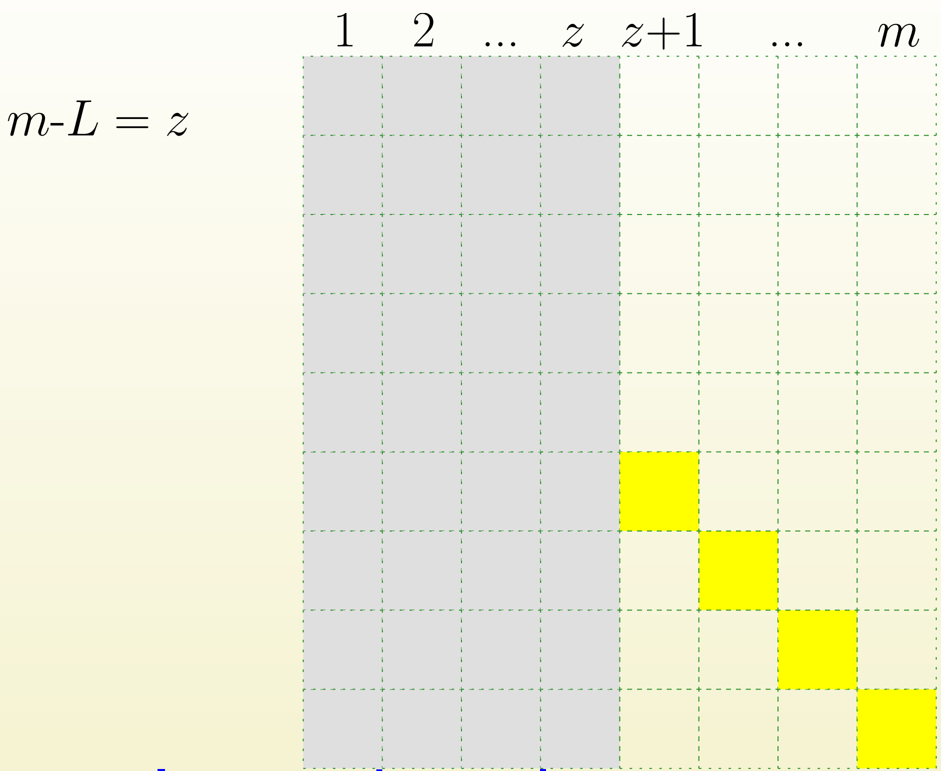

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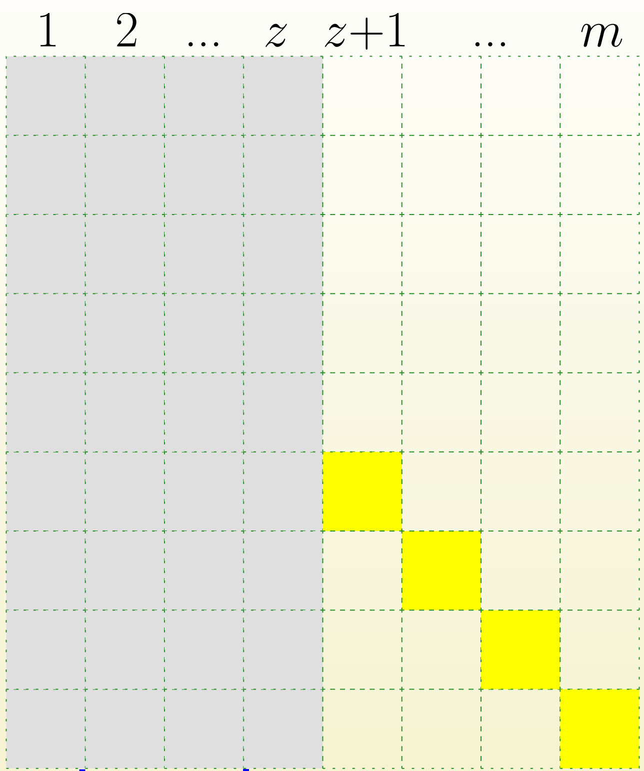

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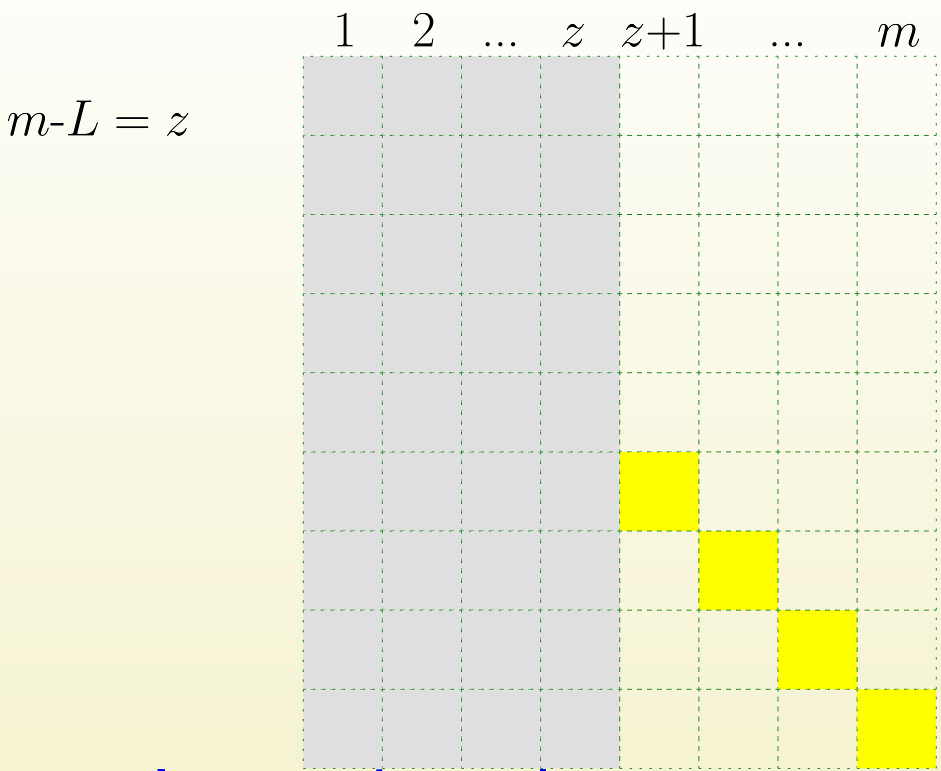

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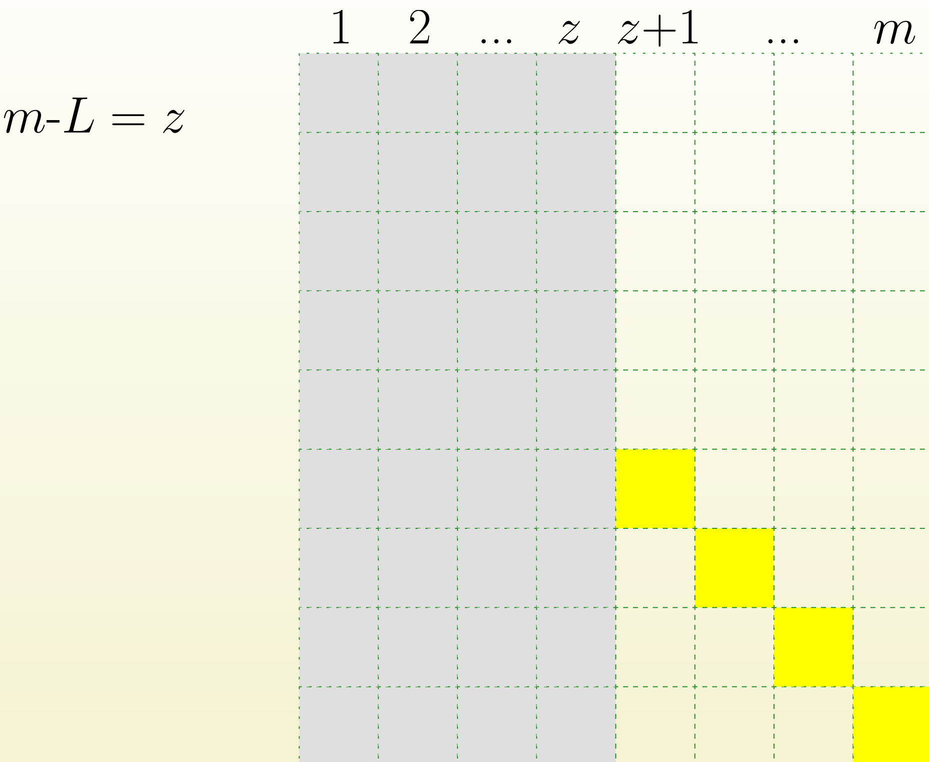

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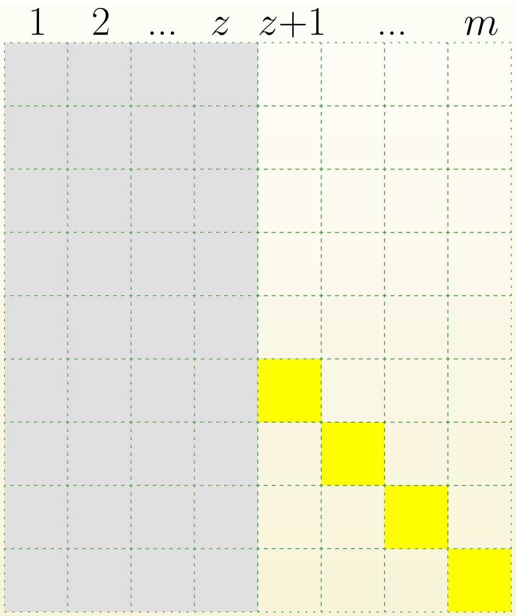
  - ▷ $block_{\#}$ for remaining rows $\leq z$+i-$i = z$

m-$L = z$



Total work $\approx z^2 + (m-z)(n-L+z+1) = \mathcal{O}(m(n-L))$

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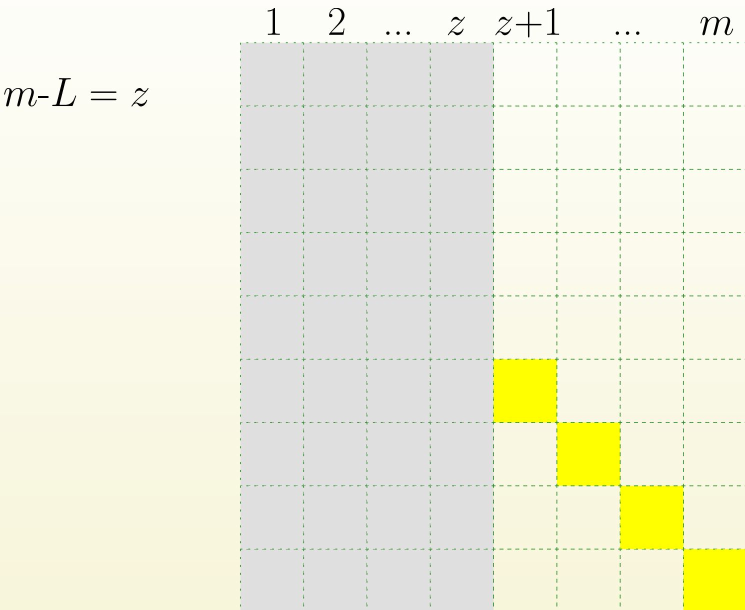
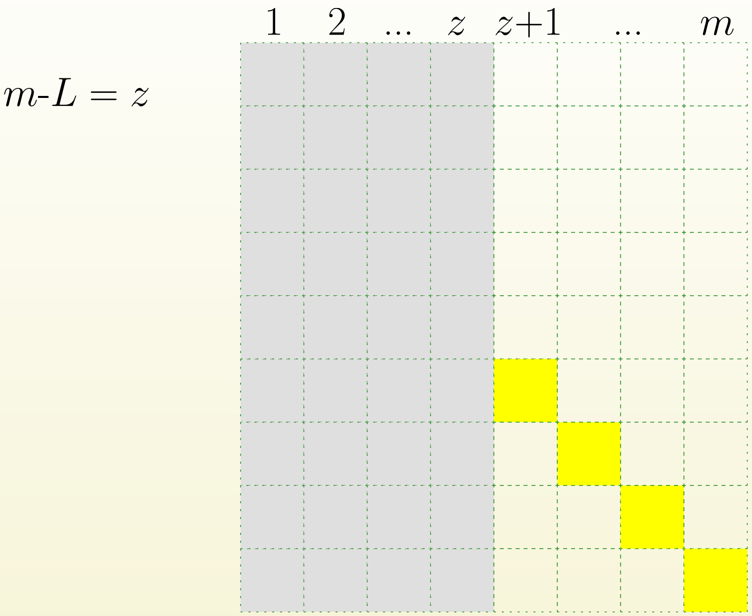
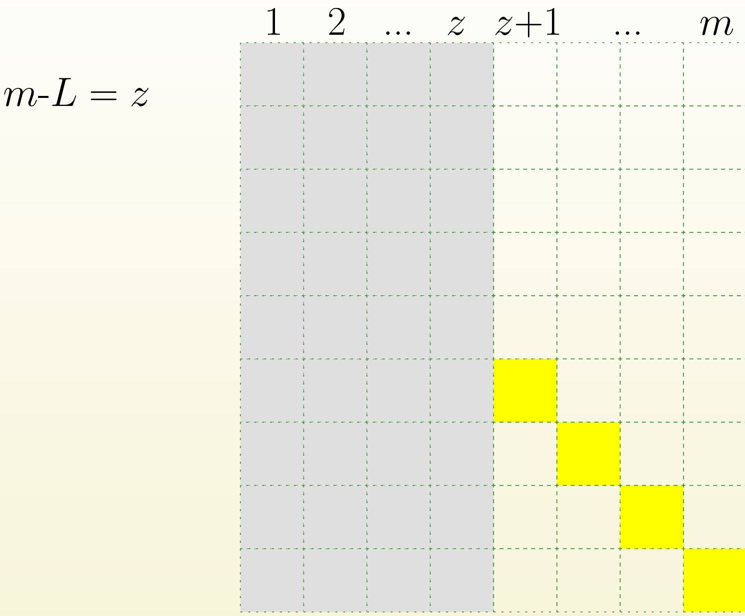


Total work $\approx z^2 + (m-z)(n-L+z+1) = \mathcal{O}(m(n-L))$,
and also bounded by $\mathcal{O}(mL)$



Total work $\approx z^2 + (m-z)(n-L+z+1) = \mathcal{O}(m(n-L))$, and also bounded by $\mathcal{O}(mL)$

Total time complexity $\mathcal{O}(\sigma n + \min\{mL, L(n-L)\})$



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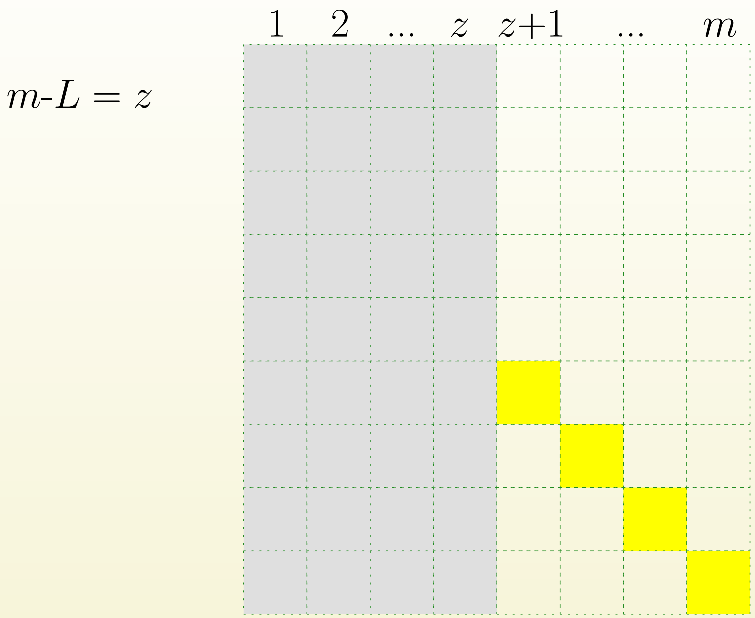
- If $L < \frac{m}{2}$, then $\mathcal{O}(mL) = \mathcal{O}(L(n-L))$



Total work $\approx z^2 + (m-z)(n-L+z+1) = \mathcal{O}(m(n-L))$,
and also bounded by $\mathcal{O}(mL)$

Total time complexity $\mathcal{O}(\sigma n + \min\{mL, L(n-L)\})$

- If $L < \frac{m}{2}$, then $\mathcal{O}(mL) = \mathcal{O}(L(n-L))$
- If $L \geq \frac{m}{2}$, then $\mathcal{O}(m(n-L)) = \mathcal{O}(L(n-L))$